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FREE VIBRATION STUDY OF ORTHOTROPIC SQUARE PLATE WITH BI-DIRECTIONAL CIRCULAR THICKNESS VARIATION UNDER THERMAL EFFECT

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ABSTRACT

Visco-elastic plates are widely utilized in engineering structures such as aerospace components, mechanical systems, and industrial applications. Accurate prediction of their vibrational behavior is essential for ensuring structural safety and efficiency. Plates with variable thickness are particularly relevant in advanced fields including nuclear engineering, marine structures, and earthquake-resistant systems.

The present study develops a mathematical model to investigate the free vibration characteristics of a visco-elastic square plate with thickness varying linearly in both spatial directions. The plate is assumed to be clamped along all edges and subjected to a two-dimensional thermal field, which varies linearly in one direction and parabolically in the other.

The Rayleigh–Ritz method is employed using a two-term deflection function to derive the governing frequency equation. Natural frequencies are computed for different values of taper parameters and thermal gradients using MATLAB. The results are presented graphically, providing useful insights for the design of structures operating under thermal and non-uniform geometric conditions.

Keywords: Plate, thickness, frequency, vibration, temperature.

INTRODUCTION

The vibration behaviour of structural elements has long been a subject of central importance in applied mechanics, particularly in the design of aerospace panels, mechanical components, and civil engineering structures. Among these elements, square plates occupy a significant position due to their frequent use in engineering applications where geometric symmetry and ease of fabrication are desirable. However, in practical situations, such plates rarely possess uniform thickness or operate under ideal thermal conditions. Instead, they often exhibit spatial variation in thickness and are exposed to temperature gradients, both of which considerably influence their dynamic response.

The present study focuses on the vibration analysis of a square plate whose thickness varies in a circular manner along both spatial directions, commonly referred to as bi-directional circular thickness variation. This type of variation represents a more realistic modelling approach for advanced engineering structures, where material distribution is intentionally modified to enhance strength, reduce weight, or improve functional efficiency. Unlike uniform plates, such non-homogeneous configurations introduce additional complexity into the governing differential equations, as stiffness and mass distribution become functions of position. Consequently, the vibrational characteristics, including natural frequencies and mode shapes, are significantly altered.

In addition to geometric variation, thermal effects play a crucial role in determining the behaviour of plate structures. When a plate is subjected to temperature changes, thermal stresses are induced due to constrained expansion or contraction. These stresses modify the effective stiffness of the plate and, therefore, influence its vibration response. In many engineering environments, such as turbine blades, electronic devices, and space structures, temperature variations are neither uniform nor negligible. A bi-directional thermal field, varying along both axes of the plate, introduces further coupling between thermal and mechanical responses, making the analysis more intricate and practically relevant.

The combined influence of circular thickness variation and thermal loading creates a complex interaction between structural rigidity, mass distribution, and thermal stresses. This interaction cannot be captured adequately by classical models that assume constant thickness and uniform temperature. Instead, refined mathematical formulations are required, often involving higher-order partial differential equations with variable coefficients. Solving such equations typically demands analytical approximation methods or numerical techniques, depending on the boundary conditions and the nature of variation involved.

From an engineering perspective, understanding these coupled effects is essential for predicting resonance conditions and preventing structural failure. Even small deviations in thickness or temperature distribution can lead to noticeable shifts in natural frequencies, potentially causing resonance under operational loads. Therefore, a detailed vibration analysis not only contributes to theoretical development but also provides practical guidelines for safer and more efficient design.

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In this context, the present work aims to investigate the free vibration characteristics of a square plate with bi-directional circular thickness variation under thermal effects. By incorporating realistic assumptions regarding thickness distribution and temperature gradients, the study seeks to bridge the gap between idealized theoretical models and real-world structural behaviour. The results obtained are expected to offer deeper insights into the dynamic performance of non-uniform plates and to support the development of optimized structural designs in thermally sensitive environments.

MATHEMATICAL FORMULATION

For a thin, orthotropic square plate of side length a, the governing equation of motion may be expressed in an expanded form as::

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = \rho h \frac{\partial^2 w}{\partial t^2} \quad (1)$$

Where

- M_x, M_y, and M_{xy}denote the bending and twisting moments per unit length of the plate.
- ρrepresents the mass density per unit volume.
- $h$is the thickness of the plate.
- $w(x, y, t)$is the transverse displacement at time t.

The bending and twisting moments are given by:

$$M_x = -\bar{D}\left[D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -\bar{D}\left[D_1 \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right]$$

$$M_{xy} = -2\bar{D}D_{xy}\frac{\partial^2 w}{\partial x\,\partial y}$$

The flexural rigidities in the principal directions are defined as:

$$D_x = \frac{E_x h^3}{12(1 - \nu_x \nu_y)}$$

$$D_y = \frac{E_y h^3}{12(1 - \nu_x \nu_y)}$$

The time-dependent function satisfies:

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0 \quad (2)$$

The solution represents harmonic motion and can be written as:

$$T(t) = A\cos(\omega t) + B\sin(\omega t) \quad (3)$$

This indicates that the plate vibrates periodically with angular frequency ω, where constants A and B depend on initial conditions.

The thickness of the plate considered in this study is assumed to vary bi-linearly along both coordinate axes. This variation may be expressed as

$$g = g_0\left[1 + \beta_1\left(1 - \frac{x}{a}\right)\right]\left[1 + \beta_2\left(1 - \frac{y}{a}\right)\right] \quad (4)$$

where $\beta_1$and $\beta_2$are tapering parameters governing the thickness variation along the x- and y-directions, respectively. When $\beta = 0$, the plate thickness reduces to a uniform value.

The temperature distribution is also assumed to vary bi-linearly along both axes, and is given by

$$\tau = \tau_0\left[\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right)\right] \quad (5)$$

Here, $a$denotes the characteristic length of the plate, while $\tau_0$represents the temperature at the origin.

Since material properties are influenced by temperature, the Young's modulus is taken as temperature-dependent and is modeled as

$$Y(\tau) = Y_0(1 - \gamma\tau) \quad (6)$$

where $Y_0$is the reference Young's modulus and $\gamma$is a constant representing the rate at which stiffness decreases with temperature.

Substituting the temperature distribution from equation (5) into equation (6), the spatial variation of Young's modulus becomes

$$Y = Y_0\left[1 - \gamma\tau_0\left\{\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right)\right\}\right]$$

For simplicity, introducing $\alpha = \gamma\tau_0$, the above expression may be written as

$$Y = Y_0\left[1 - \alpha\left\{\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right)\right\}\right] \quad (7)$$

This formulation shows that the stiffness of the plate decreases progressively away from the origin, following the same spatial pattern as the imposed temperature field.

The parameter $\alpha = \gamma\tau_0$represents the thermal gradient within the plate, where its value ranges between 0 and 1 ($0 \le \alpha \le 1$). In the case of a homogeneous material, no spatial variation is assumed in Poisson's ratio; hence, the circular variation parameter is taken as zero, i.e.,

$$h = 0 \quad (7)$$

Under these assumptions, the plate satisfies simply supported boundary conditions along all edges. Accordingly, both the deflection function and its slope vanish at the boundaries, which may be expressed as:

$$\varphi = \frac{\partial\varphi}{\partial x} = 0 \text{ at } x = 0, a$$

$$\varphi = \frac{\partial\varphi}{\partial y} = 0 \text{ at } y = 0, a$$

To represent the deformation of the plate, an admissible deflection function is selected in a polynomial form that inherently satisfies these boundary conditions. It is written as:

$$\varphi = \left[\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right)\right]^2\left[B_1 + B_2\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right)\right] \quad (8)$$

where $B_1$and $B_2$are arbitrary constants determined through an energy minimization approach.

The Rayleigh–Ritz method is employed to evaluate the natural frequency of vibration. The underlying principle of this method is that, at resonance, the maximum strain energy equals the maximum kinetic energy. This condition leads to the variational statement:

$$\delta(PE - KE) = 0 \quad (9)$$

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The total strain energy of the plate, considering bending effects, is expressed as:

$$PE = \frac{1}{2}\int_0^1\int_0^1 D_1\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\varphi}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2\varphi}{\partial x\,\partial y}\right)^2\right]dy\,dx \quad (10)$$

Similarly, the kinetic energy associated with the vibrating plate is given by:

$$KE = \frac{1}{2}\rho h\omega^2\int_0^1\int_0^1 g\ \varphi^2\,dy\,dx \qquad (11)$$

These expressions form the basis for determining the frequency parameters by substituting the assumed deflection function into the energy equations and applying the Rayleigh–Ritz procedure.

Using Equations (8) and (11), the flexural rigidity of the plate can be expressed as

$$D_1 = \frac{Y_o g_o^3}{12(1-\nu^2)}[1-\alpha_1(1-x/a)(1-y/a)][1+\beta_1(1-(1-x/a))][1+\beta_2(1-(1-y/a))]^3 \qquad (12)$$

To simplify the formulation, non-dimensional variables are introduced for convenience as

$$X = \frac{x}{a}, Y = \frac{y}{a} \qquad (13)$$

Substituting Equation (13) into Equations (10) and (11), the expression transforms into the following non-dimensional form:

$$P_S^* = \frac{Y_o g_o^3}{24a^2(1-\nu^2)}\int_0^1\int_0^1 [1-\alpha_1(1-X)(1-Y)][1+\beta_1(1-(1-X))][1+\beta_2(1-(1-Y))]^3\left[\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial Y^2}+2\nu\frac{\partial^2\phi}{\partial X^2}\frac{\partial^2\phi}{\partial Y^2}+2(1-\nu)\left(\frac{\partial^2\phi}{\partial X\,\partial Y}\right)^2\right]dX\,dY \qquad (14)$$

The transformation into dimensionless coordinates plays a significant role in simplifying the mathematical formulation of plate vibration problems. By removing dependence on physical dimensions such as length and thickness, the governing equations become more general in nature. This allows the results to be applied to a wide range of plate geometries without repeating the entire analysis, which is particularly useful in vibration and stability studies.

The dimensionless strain energy functional can be expressed as:

$$K_S^* = \frac{1}{2}k^2\int_0^1\int_0^1 [(1+\beta_1(1-X))(1+\beta_2(1-Y))\,\phi^2]\,dXdY \qquad (15)$$

By substituting the assumed deflection function into the governing functional expressions, and using equations (9), (14), and (15), the system reduces to the following frequency equation:

$$P_S^* - \lambda^2 K_S^* = 0 \qquad (16)$$

Here, the parameter λ^2represents the non-dimensional frequency parameter, given by:

$$\lambda^2 = \frac{12(1-\nu^2)\rho k^2 a^4}{Eh^3}$$

This parameter effectively captures the combined influence of material properties (Young's modulus E, density ρ, Poisson's ratio ν) and geometric characteristics (plate thickness h, length a) on the vibration behavior.

Since the assumed deflection function ϕcontains unknown constants $B_1$and B_2, the application of the Ritz method requires minimizing the functional with respect to these constants. This leads to the following set of stationary conditions:

$$\frac{\partial}{\partial B_i}(P_S^* - \lambda^2 K_S^*) = 0, i = 1,2 \qquad (17)$$

On simplification, these conditions yield a system of homogeneous linear equations:

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{bmatrix}\begin{bmatrix} B_1 \\ B_2\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix} \qquad (18)$$

This system admits a non-trivial solution only when the determinant of the coefficient matrix vanishes. Therefore, setting the determinant equal to zero leads to the characteristic equation, from which the natural frequency parameters of the plate can be obtained.

Where d_{11}, $d_{12} = d_{21}$ and d_{22} involve parametric constants and frequency parameter. To get a non-trivial solution, the determinant of the coefficient matrix of equation (18) must be zero. So, we get equation of frequency as

$$\begin{vmatrix} d^{11} & d^{11} \\ d^{21} & d^{22}\end{vmatrix} = 0 \qquad (24)$$

RESULT AND DISCUSSION

A quadratic expression in the frequency parameter λ^2 is obtained by using Equation (23). The first two natural frequencies of vibration of the clamped square plate, labelled as λ_1 (Mode 1) and λ_2 (Mode 2), are found by solving this quadratic. These frequency parameters have been assessed for several taper constant combinations. The thermal gradient α and taper constant β_1 & β_2. MATLAB and other advanced computational tools have been used for all numerical calculations. The findings show how the heat distribution and the geometric taper affect the plate's vibrational behaviour. Vibrational frequency modes for homogeneous and tapered square plate are calculated for different values of temperature gradient α and taper constant (β_1 & β_2). All the calculation are presented in the form of tables.

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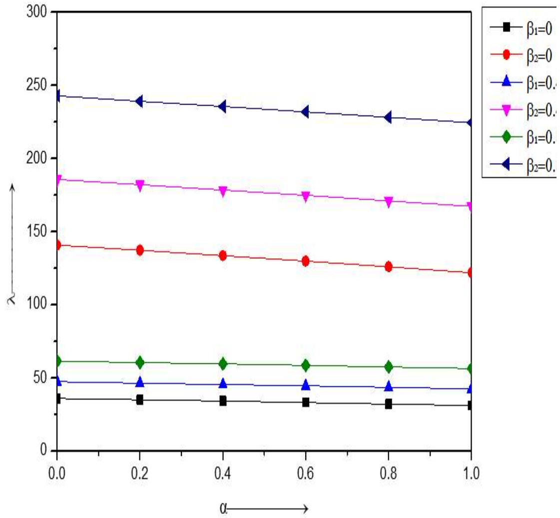
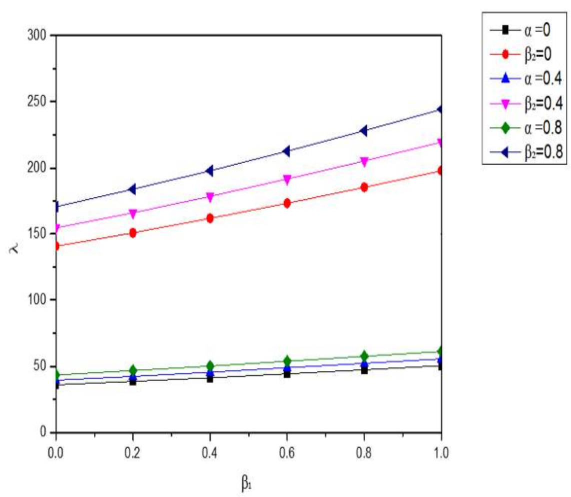
In Graph 2.1: - In this fig. we can see that the value of frequency decreasing in both the modes of vibration when we increasing value of thermal effect α from 0.0 to 1.0. for $\beta_1 = \beta_2 = 0$, $\beta_1 = \beta_2 = 0.4$ & $\beta_1 = \beta_2 = 0.8$ for both mode of vibrations.



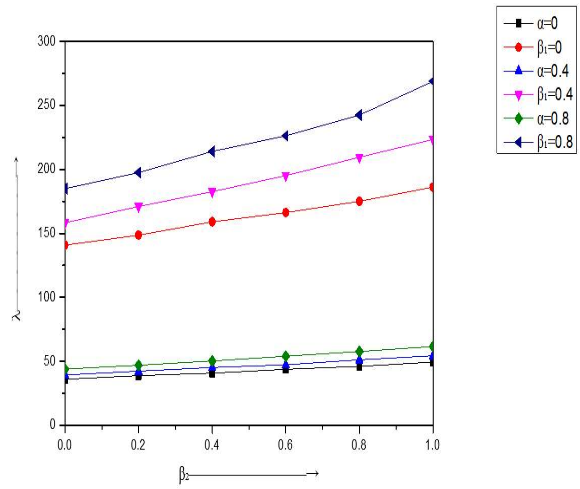
Graph 2.1 Frequency Vs Thermal Gradient (α)

Graph 2.2 :- In this fig. we can see that the value of frequency increasing in both the modes of vibration when we increasing value of taper constant β_1 from 0.0 to 1.0 for $\alpha = \beta_2 = 0$, $\alpha = \beta_2 = 0.4$ & $\alpha = \beta_2 = 0.8$ for both mode of vibrations.



Graph 2.2 Frequency Vs Taper Constant (β₁)

Graph 2.3:- In this fig. we can see that the value of frequency increasing in both the modes of vibration when we increasing value of taper constant β_2 from 0.0 to 1.0 for $\alpha = \beta_1 = 0$, $\alpha = \beta_1 = 0.4$ & $\alpha = \beta_1 = 0.8$ for both mode of vibrations.



Graph 2.3 Frequency Vs Taper Constant (β₂)

CONCLUSION

This study examines the free vibration behaviour of a clamped square plate whose thickness changes in a circular manner in both directions, while also being affected by temperature variation. The results clearly show that both temperature and thickness variation have a strong influence on the vibration of the plate.

From the graphs, it is observed that when the thermal parameter α increases from 0.0 to 1.0, the natural frequency decreases in both modes of vibration. This happens for all values of taper constants ($\beta_1 = \beta_2 = 0$, 0.4, and 0.8). The reason is that an increase in temperature reduces the stiffness of the plate, making it more flexible and lowering its vibration frequency.

On the other hand, when the taper constants β_1 and β_2 increase from 0.0 to 1.0, the frequency increases in both modes. This is seen for all values of α. Increasing β_1 or β_2 means the plate becomes thicker in certain regions, which increases its stiffness and results in higher frequencies.

In conclusion, temperature and thickness variation have opposite effects on vibration. Temperature reduces frequency, while thickness variation increases it. These results are helpful for designing structures where both heat and vibration are important factors.

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