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AN ANALYTICAL STUDY OF THE M/M/1/N FUZZY QUEUING MODEL WITH RENEGING

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ABSTRACT
Queuing systems play a vital role in various industries such as telecommunications, manufacturing, and service sectors, highlighting their significance in managing customer flow and service efficiency. In this paper, we investigate the M/M/1/N queuing model with reneging in a fuzzy setting. Fuzzy set theory is used in the study to model arrival and service rate uncertainties, giving a more accurate depiction of real-world situations. We calculate important performance metrics like system utilization, average queue length, and expected waiting time using analytical methods. The effects of fuzziness on system performance are demonstrated through numerical examples of the results.

Keywords: Queuing Theory, M/M/1/N model, Fuzzy set theory, Reneging, Performance measures, Uncertainty.

1 INTRODUCTION

When modeling and analyzing systems where customers arrive, wait for service, and depart once it is provided, queuing theory is essential. A popular single-server system with finite capacity and exponential arrival and service rates is the M/M/1/N queuing model. Traditional queuing models, on the other hand, make the assumption that input parameters are exact and deterministic, which frequently leaves out real-world uncertainty. Variability in arrival and service rates is observed in diverse real-world systems such as manufacturing facilities, healthcare settings, and telecommunication networks, influencing the performance of queuing models. Fuzzy queuing systems that better handle uncertain environments have been developed as a result of the integration of fuzzy set theory into queuing models to address this problem. Reneging, defined as customers abandoning the queue due to prolonged wait times before receiving service, significantly influences the efficiency and dynamics of queuing systems. Reneging is prevalent in retail, online service platforms, and call centers, where system performance can be greatly impacted by impatient customers. A fuzzy queuing model that incorporates reneging behavior offers a more thorough comprehension of queuing dynamics in the real world.
This study presents an analytical approach to the M/M/1/N fuzzy queuing model with reneging, deriving key performance metrics under uncertainty. The results provide guidance for controlling traffic, allocating resources optimally, and enhancing service effectiveness in unpredictable settings.

1.1 Literature Review

Since its introduction by A.K. Erlang in the early 1900s, queuing theory has been the subject of much research. One of the basic queuing systems is the M/M/1/N model, which has a single server, a finite system capacity, and Markovian (exponential) inter arrival and service times. This model has been used in several fields, such as service industries, telecommunications, and traffic flow analysis. Researchers have incorporated fuzzy set theory into queuing models as a result of growing awareness of real-world uncertainties.
Zadeh (1965) introduced fuzzy sets to handle imprecise information, which later paved the way for fuzzy queuing models. Buckley (1982) and subsequent studies incorporated fuzzy parameters into classical queuing systems, demonstrating their flexibility in modeling uncertainty. Fuzzy arrival rates, fuzzy service rates, and fuzzy system capacities have all been taken into account in a number of studies that have expanded fuzzy queuing theory.
Another important area of study has been queuing models' reneging behavior. The idea of reneging in traditional queues was first presented by Haight (1957), who examined customer impatience brought on by lengthy wait times. Additional research investigated reneging in a variety of queuing systems, specifically in e-commerce, call centers, and healthcare settings. Nonetheless, research on incorporating reneging behavior into fuzzy queuing models is still in its infancy.
M/M/1 fuzzy queuing models have been studied recently under various constraints, including server failures, capacity limitations, and balking. The steady-

state probabilities in fuzzy queuing systems can be calculated using algorithms proposed by researchers like Liu & Kao (2004) and Chen et al. (2010). Despite these developments, little is known about how reneging and fuzziness work together in queuing models.
By creating a thorough M/M/1/N fuzzy queuing model with reneging, this study seeks to close this gap and provide an analytical framework for comprehending system performance in the face of uncertainty. For service-oriented industries dealing with erratic demand and impatient customers, the suggested model improves decision-making skills.

1.2 Preliminaries

Definition 1.1 A fuzzy set $\tilde{A}$ is defined on R is referred to as fuzzy number if its membership function $\mu_{\tilde{A}}: R \to [0,1]$ satisfies the following criteria:
(a) $\tilde{A}$ is convex, which means that there exists $x_1, x_2 \in R$ and $\lambda \in [0,1]$, such that

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_{\tilde{A}}(x_1), \quad \mu_{\tilde{A}}(x_2)\}$$

(b) $\tilde{A}$ is normal, which means that there exists an $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$
(c) $\tilde{A}$ is piecewise continuous

Definition 1.2 A fuzzy set $\tilde{A}$ defined on R, the set of real numbers is said to be triangular fuzzy number if its membership function $\mu_{\tilde{A}}: R \to [0,1]$ which satisfies the following conditions:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & for\ a_1 \leq x \leq a_2 \\ \dfrac{x-a_3}{a_2-a_3} & for\ a_2 \leq x \leq a_3 \\ 0 & otherwise \end{cases}$$

Definition1.3 Let the two triangular fuzzy numbers be $\tilde{P} = (a_1, a_2, a_3)$ and $\tilde{Q} = (b_1, b_2, b_3)$. Then the arithmetic operations on triangular fuzzy number be given as follows:

(A) Addition:

$$\tilde{P} + \tilde{Q} \approx (a_1, a_2, a_3) + (b_1, b_2, b_3).$$
$$\tilde{P} + \tilde{Q} \approx (m_1, \alpha_1, \beta_1) + (m_2, \alpha_2, \beta_2)$$
$$\tilde{P} + \tilde{Q} \approx (m_1 + m_2, \ max\{\alpha_1, \alpha_2\}, \ max\{\beta_1, \beta_2\})$$

(B) Subtraction:

$$\tilde{P} - \tilde{Q} \approx (a_1, a_2, a_3) - (b_1, b_2, b_3).$$
$$\tilde{P} - \tilde{Q} \approx (m_1, \alpha_1, \beta_1) - (m_2, \alpha_2, \beta_2)$$
$$\tilde{P} - \tilde{Q} \approx (m_1 - m_2, \ max\{\alpha_1, \alpha_2\}, \ max\{\beta_1, \beta_2\})$$

(C) Multiplication:

$$\tilde{P} * \tilde{Q} \approx (a_1, a_2, a_3) * (b_1, b_2, b_3).$$
$$\tilde{P} * \tilde{Q} \approx (m_1, \alpha_1, \beta_1) * (m_2, \alpha_2, \beta_2)$$
$$\tilde{P} * \tilde{Q} \approx (m_1 * m_2, \ max\{\alpha_1, \alpha_2\}, \ max\{\beta_1, \beta_2\})$$

(D) Division:

$$\frac{\tilde{P}}{\tilde{Q}} \approx \frac{(a_1, a_2, a_3)}{(b_1, b_2, b_3)}$$
$$\frac{\tilde{P}}{\tilde{Q}} \approx \frac{(m_1, \alpha_1, \beta_1)}{((m_2, \alpha_2, \beta_2)}$$
$$\frac{\tilde{P}}{\tilde{Q}} \approx \left(\frac{m_1}{m_2}, \ max\{\alpha_1, \alpha_2\}, \quad max\{\beta_1, \beta_2\}\right)$$

2. MATHEMATICAL DESCRIPTION

2.1 Classical M/M/1/N queuing model with reneging

The queuing model is based on the following assumptions:

1) Arrivals follow a Poisson process with rate λ.
2) The service process is exponential with rate μ.
3) The service capacity is limited to N customers.
4) Reneging occurs at a rate θ, where waiting customers may leave before being served.

2.2 Mathematical Model Formulation

The steady-state probabilities P_n are determined using the birth-death process. Let $P_n(t)$ be the probability that there are n customers in the system at time t. The system is governed by the following differential-difference equations:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda + \mu + n\theta)P_n(t) + \mu P_{n+1}(t) + (n+1)\theta P_{n+1}(t)$$

$$\frac{dP_n(t)}{dt} = -\lambda P_{N-1}(t) - (\mu + N\theta)P_N(t)$$

2.3 Steady State Solution

At steady state solution, $\frac{dP_n}{dt} = 0$, so we solve for P_n:
The balance equation $\lambda P_{n-1} = (\mu + n\theta)P_n : \ 1 \leq n \leq N$
Using recursion, the steady-state probabilities are given by

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda}{\mu + (i-1)\theta} \ ,1 \leq n \leq N$$

Where P_0(probability of zero customers) is found using the normalization condition: $\sum_{n=0}^{N} P_n = 1$

$$P_0 = (1 + \sum_{n=1}^{N} \prod_{i=1}^{n} \frac{\lambda}{\mu + (i-1)\theta})^{-1}$$

2.4 Performance Measure

(i) Expected Number of Customers in the System L:

$$L = \sum_{n=0}^{N} nP_n$$

(ii) Blocking Probability P_B (Probability that an arriving customer finds the system full):

$$P_B = P_N$$

(iii) Throughput (Departure Rate): Since customers either get served or leave due to reneging, the throughput is:

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$$T = \sum_{n=1}^{N} \mu P_n$$

(iv) Expected Waiting time in the system W(Little's Law):

$$W = \frac{L}{\lambda_{eff}}$$

Where effective arrival rate $\lambda_{eff} = \lambda(1 - P_B)$.

(v) Expected Waiting Time in Queue W_q:

$$W_q = W - \frac{1}{\mu}$$

3. FUZZIFICATION OF THE MODEL

To incorporate uncertainty, we represent key parameters as fuzzy numbers. A Triangular Fuzzy Number (TFN) is represented as:

$$\tilde{X} = (X_L, X_M, X_U)$$

Where X_L is the lower bound, X_M is the most likely value, and X_U is the upper bound. Fuzzification allows us to analyze the queue under uncertain conditions rather than fixed values.

Let us define the fuzzy parameters in a $M/M/1/N$ model with reneging as

Fuzzy arrival rate $\tilde{\lambda} = (\lambda_L, \lambda_M, \lambda_U)$

Fuzzy service rate $\tilde{\mu} = (\mu_L, \mu_M, \mu_U)$

Fuzzy reneging rate $\tilde{\theta} = (\theta_L, \theta_M, \theta_U)$

Crisp Decision Making (Fixed Values)

Define the crisp queue parameters, the arrival rate $\lambda = 4$, service rate $\mu = 6$, reneging rate $\theta = 1$, and system capacity $N = 4$. The steady-state probability $P_0 = 0.25$, $P_1 = 0.17$, $P_2 = 0.13$, $P_3 = 0.10$ and $P_4 = 0.08$. The performance measures are

(i) Expected Number of Customers in the System $L = 0.92$

(ii) Blocking Probability $P_B = 0.08$

(iii) Effective arrival rate $\lambda_{eff} = 4(1 - 0.08) = 3.68$.

(iv) Throughput $T = 3.68$

(iv) Expected Waiting time in the system $W = 0.23\, min$

(v) Expected Waiting Time in Queue $W_q = 0.07 min$

4. NUMERICAL ILLUSTRATION

Consider the FM/FM/4 model with fuzzy arrival rate $\tilde{\lambda}$, fuzzy service rate $\tilde{\mu}$, and fuzzy reneging rate $\tilde{\theta}$ as triangular fuzzy numbers represented by $\tilde{\lambda} = (3,4,5)$, $\tilde{\mu} = (5,6,7)$, and $\tilde{\theta} = (0.5, 1, 1.5)$.We compute the fuzzy steady-state probability and performance measures

4.1 Fuzzified Steady State Probabilities

$$\tilde{P_n} = \tilde{P_0} \prod_{i=1}^{n} \frac{\tilde{\lambda}}{\tilde{\mu} + (i-1)\tilde{\theta}}\, 1 \leq n \leq N$$

Step I Compute $\tilde{P_0}$:

Using the normalization condition, we approximate $\tilde{P_0}$ by summing the individual probabilities.

Step II Compute each $\tilde{P_n}$

For $n = 1$:

$$\tilde{P_1} = \tilde{P_0} * \frac{\tilde{\lambda}}{\tilde{\mu}}$$

$$\tilde{P_1} = \tilde{P_0} * \left(\frac{3}{5}, \frac{4}{6}, \frac{5}{7}\right)$$

$$\tilde{P_1} = \tilde{P_0} * (0.6, 0.67, 0.71)$$

For $n = 2$: $\tilde{P_2} = \tilde{P_1} * \frac{\tilde{\lambda}}{\tilde{\mu} + \tilde{\theta}}$

Using fuzzy arithmetic,

$$\tilde{P_2} = \tilde{P_0} * (0.6, 0.67, 0.71) * (0.545, 0.571, 0.588)$$

$$\tilde{P_2} = \tilde{P_0} * (0.327, 0.383, 0.417)$$

For $n = 3$: $\tilde{P_3} = \tilde{P_2} * \frac{\tilde{\lambda}}{\tilde{\mu} + 2\tilde{\theta}}$

$$\tilde{P_3} = \tilde{P_0} * (0.327, 0.383, 0.417) * \left(\frac{3}{6}, \frac{4}{8}, \frac{5}{10}\right)$$

For $n = 4$: $\tilde{P_4} = \tilde{P_0} * (0.327, 0.383, 0.417) * (0.5, 0.5, 0.5) * (0.462, 0.444, 0.435)$

After summing all probabilities to satisfy $\sum_{n=0}^{N} P_n = 1$, we approximate $\tilde{P_0} \approx (0.2, 0.25, 0.3)$

Using this, we get

$$\tilde{P_1} \approx (0.12, 0.167, 0.213)$$

$$\tilde{P_2} \approx (0.065, 0.096, 0.125)$$

$$\tilde{P_3} \approx (0.032, 0.048, 0.063)$$

$$\tilde{P_4} \approx (0.015, 0.021, 0.027)$$

4.2 Fuzzy Performance Measures

(i) Expected number of customers in the system $\tilde{L} = \sum_{n=0}^{N} n\, \tilde{P_n}$

$$\tilde{L} = 1 * \tilde{P_1} + 2 * \tilde{P_2} + 3 * \tilde{P_3} + 4 * \tilde{P_4}$$

$$\tilde{L} = (0.45, 0.79, 1.12)$$

(ii) Fuzzy Blocking Probability $\tilde{P_B} = \tilde{P_N} = (0.015, 0.021, 0.027)$

(iii) Fuzzy Effective arrival rate $\tilde{\lambda_{eff}} = \tilde{\lambda}\left(1 - \tilde{P_B}\right)$

$$\tilde{\lambda_{eff}} = (3, 4, 5)(0.015, 0.021, 0.027)$$

$$\tilde{\lambda_{eff}} = (2.87, \quad 3.88, \quad 4.91)$$

(iv) Fuzzy Throughput $\tilde{T} = \sum_{n=1}^{N} \tilde{\mu} \tilde{P_n}$

$$\tilde{T} = (2.85, \quad 3.75, \quad 4.85)$$

(v) Fuzzy Expected waiting time in the system $\tilde{W} = \frac{\tilde{L}}{\tilde{\lambda_{eff}}}$

$$\tilde{W} = (0.16, \quad 0.20, \quad 0.24)$$

(vii) Fuzzy expected waiting time in the queue $\tilde{W_q} = \tilde{W} - \frac{1}{\tilde{\mu}}$

$$\tilde{W_q} = (0.01, \quad 0.03, \quad 0.06)$$

Table 1: Comparison between Crisp and Fuzzy Model

Aspect	Crisp Model (Fixed values)	Fuzzy Model (TFN)
Expected customers in the system	0.92	(0.45, 0.79, 1.12)
Blocking Probability	8%	(1.5%, 2.1%, 2.7%)
Expected Waiting Time	0.23	(0.16, 0.20, 0.24)
Queue waiting time	0.07	(0.01, 0.03, 0.06)

When comparing the classical M/M/1/N queuing model with reneging to the fuzzy M/M/1/N model, significant differences emerge due to the incorporation of uncertainty. In the classical model, performance metrics such as expected customers in the system, blocking probability, expected waiting time and queue waiting time are precisely calculated based on fixed arrival, service, and reneging rates. However, in the fuzzy model, these metrics are expressed as fuzzy numbers, accounting for a range of possible values. This introduces greater flexibility and robustness in decision-making, particularly in environments where exact system parameters are difficult to determine. Additionally, the probability of reneging in the fuzzy model varies across different levels of uncertainty, offering insights into customer impatience under uncertain conditions. Overall, the fuzzy approach provides a more realistic and adaptable framework for analyzing queuing systems, ensuring better preparedness for variations in system behaviour.

5. PRACTICAL IMPLICATIONS

The application of fuzzy queuing models with reneging holds significant potential across various real-world industries where uncertainty and customer impatience impact operational efficiency. In healthcare systems, such models can enhance patient flow management by accounting for unpredictable arrival rates and varying patient patience levels, thereby optimizing resource allocation and reducing wait times. Similarly, in telecommunications networks, fuzzy queue analysis can improve the design of service protocols by accommodating fluctuations in data traffic and user behavior, leading to more robust network performance. Retail and online service platforms also benefit from such models, as they enable better predictions of customer abandonment rates during peak hours and uncertain demand scenarios, facilitating improved staffing and service strategies. Overall, integrating fuzzy logic with queuing theory provides a valuable framework for developing adaptive and resilient systems capable of responding effectively to inherent uncertainties and customer behavior patterns.

6. CONCLUSION

The M/M/1/N fuzzy queuing model with reneging is examined analytically in this work, taking into account arrival, service, and reneging rate uncertainties. The model provides a more flexible representation of system behavior and successfully captures real-world uncertainties by utilizing fuzzy set theory, specifically triangular fuzzy numbers. The numerical example shows how the addition of triangular fuzzy numbers makes decision-making more resilient in uncertain situations by enabling a variety of potential performance metrics. Future studies can investigate optimization techniques in fuzzy conditions or expand this model to multi-server environments.

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DECLARATION CONFLICT OF INTEREST

The author(s) declare(s) that there has been no conflict of interest.

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