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A Characterization of Fuzzy Completely Disconnected Spaces

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Abstract- In this paper, several characterizations of fuzzy perfectly disconnected spaces, are established. The conditions under which fuzzy perfectly disconnected spaces become fuzzy Baire spaces, fuzzy σ-Baire spaces and fuzzy almost irresolvable spaces , are also obtained.

Keywords: Fuzzy dense set, fuzzy residual set, fuzzy simply open set, fuzzy G_δ -set , fuzzy regular closed set.

1. INTRODUCTION

In 1965, **L.A. ZADEH** [17] introduced the concept of fuzzy sets as a new approach for modelling uncertainties. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. Topology provided the most natural framework for the concepts of fuzzy sets to flourish. In 1968, the concept of fuzzy topological space was introduced by **C. L . CHANG** [4] . The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

The concept of perfectly disconnected spaces in classical topology was defined and studied by **ERIC K. VAN DOUWEN** [16]. In the recent years, there has been a growing trend among many fuzzy topologists to introduce and study different types fuzzy topological spaces . In [13] the notion of fuzzy perfectly disconnected spaces , was introduced and the fuzzy perfectly disconnectedness continuation of this work, several characterizations of fuzzy perfectly disconnected spaces are established in this paper. It is established that in fuzzy perfectly disconnected spaces , 0_X and 1_X are the only two fuzzy simply open sets and the fuzzy interiors of fuzzy pre-closed sets are fuzzy regular closed sets . The conditions under which fuzzy perfectly disconnected spaces become fuzzy Baire spaces, fuzzy σ-Baire spaces and fuzzy almost irresolvable spaces, are also established.

2. PRELIMINARIES

In order to make the exposition self-contained, some basic notions and results used in the sequel, are given . In this work by (X,T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0,1]. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 [4] : Let (X,T) be a fuzzy topological space and λ be any fuzzy set in (X,T). The interior and the closure of λ , are defined respectively as follows :

(i). $\text{Int}(\lambda) = \vee \{ \mu / \mu \le \lambda , \mu \in T\}$ and (ii). $\text{Cl}(\lambda) = \wedge \{ \mu / \lambda \le \mu , 1-\mu \in T \}$.

Lemma 2.1 [1] : For a fuzzy set λ of a fuzzy topological space X, (i). $1 - \text{Int}(\lambda) = \text{Cl}(1 - \lambda)$ and (ii). $1 - \text{Cl}(\lambda) = \text{Int}(1 - \lambda)$.

Definition 2.2 : A fuzzy set λ in a fuzzy topological space (X,T) is called

(1) fuzzy pre-open if $\lambda \le \text{int}\,\text{cl}(\lambda)$ and fuzzy pre-closed if $\text{cl}\,\text{int}(\lambda) \le \lambda$. **[7]**

(2) fuzzy semi-open if $\lambda \le \text{cl}\,\text{int}(\lambda)$ and fuzzy semi-closed if $\text{int}\,\text{cl}(\lambda) \le \lambda$. **[1]**

(3) fuzzy β-open if $\lambda \le \text{cl}\,\text{int}\,\text{cl}(\lambda)$ and fuzzy β-closed if $\text{int}\,\text{cl}\,\text{int}(\lambda) \le \lambda$. **[3]**

(4) fuzzy regular-open if $\lambda = \text{int}\,\text{cl}(\lambda)$ and fuzzy regular-closed if $\lambda = \text{cl}\,\text{int}(\lambda)$. **[1]**

Definition 2.3 : A fuzzy set λ in a fuzzy topological space (X,T), is called

(i). ***a fuzzy dense set*** if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$. That is, $\text{cl}(\lambda) = 1$, in (X,T) [**8**] .

(ii). ***a fuzzy nowhere dense set*** if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < \text{cl}(\lambda)$. That is, $\text{int}\,\text{cl}(\lambda) = 0$, in (X,T) [**8**].

(iii). ***a fuzzy first category set*** if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of ***fuzzy second category*** [**8**] .

(iv). ***a fuzzy σ-nowhere dense set*** if λ is a fuzzy F_σ-set in (X, T) such that $\text{Int}(\lambda) = 0$ [**6**].

(v). ***a fuzzy simply open set*** if Bd (λ) is a fuzzy

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Vol. 4, No. 1, March-2019

nowhere dense set in (X,T). That is, λ is a fuzzy simply open set in (X,T) if [cl (λ) ∧ cl (1− λ)] , is a fuzzy nowhere dense set in (X,T) [12] .

(vi). ***a fuzzy somewhere dense set*** if int cl (λ) ≠ 0 in (X,T) [**9**].

(vii).***a fuzzy σ-boundary set*** if $\lambda = \vee_{i=1}^{\infty} (\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and (λ_i)'s are fuzzy regular open sets in (X, T) [**5**] .

Lemma 2.2 [1] : In a fuzzy topological space

(a). The closure of a fuzzy open set is a fuzzy regular closed set. (b). The interior of a fuzzy closed set is a fuzzy regular open set.

Theorem 2.1 [13] : : If (X, T) is a fuzzy perfectly disconnected space, then (i). The closure of a fuzzy semi-open set in (X,T) is a fuzzy open set in (X,T). (ii). The closure of a fuzzy pre-open set in (X,T) is a fuzzy open set in (X,T).

Theorem 2.2 [11] : If $cl [\wedge_{i=1}^{\infty} (\lambda_i)] = 1$, where (λ_i) 's are fuzzy dense and open sets in a fuzzy topological space (X,T), then (X, T) is a fuzzy Baire space.(λ_i)]

Theorem 2.3 [5] : If λ is a fuzzy residual set in a fuzzy topological

Theorem 2.4 [13] : If λ is a fuzzy nowhere dense set in a fuzzy perfectly disconnected space (X, T), then λ = 0, in (X, T).

Theorem 2.5 [14]: If λ is a fuzzy somewhere dense set in a fuzzy topological space (X, T), then there exist a fuzzy regular closed set η in (X, T) such that η ≤ cl (λ).

Theorem 2.6 [13] : If λ is a fuzzy regular closed set in a fuzzy perfectly disconnected space (X, T), then λ is a fuzzy open set in (X, T).

Theorem 2.7 [14] : If λ is a non-zero fuzzy β -open set in a fuzzy topological space (X, T), then λ is a fuzzy somewhere dense set in (X,T).

Theorem 2.8 [5] : : If λ is a fuzzy σ –nowhere dense set in a fuzzy topological space (X, T) , then 1 − λ is a fuzzy residual set in (X, T).

Theorem 2.9 [5] **:** If λ is a fuzzy σ-boundary set in a fuzzy topological space (X,T), then $\lambda = \vee_{i=1}^{\infty} (\mu_i)$, where (μ_i)' s are fuzzy pre-closed sets in (X,T).

Theorem 2.10 [5] **:** If λ is a fuzzy residual set in a fuzzy second category but not a fuzzy Baire space (X, T), then there exists a fuzzy closed and fuzzy residual set η in (X, T) such that cl (λ) ≤ η .

Theorem 2.11 [**5**]: If λ is a fuzzy co- σ-boundary set in (X, T), then 1 − λ is a fuzzy σ-boundary set in (X, T).

Theorem 2.12 [**6**]: In a fuzzy topological space (X, T), a fuzzy set λ is a fuzzy σ-nowhere dense set in (X,T) if and only if 1 − λ is a fuzzy dense and fuzzy G_δ-set in (X, T).

Theorem 2.13 [**6**] : If the fuzzy topological space (X, T) is a fuzzy σ-Baire and fuzzy open hereditarily irresolvable space, then (X, T) is a fuzzy Baire space.

Theorem 2.14 [15]: If a fuzzy topological space (X, T) is a fuzzy submaximal space, then (X ,T) is a fuzzy open hereditarily irresolvable space.

Theorem 2.15 [15]: If the fuzzy topological space (X, T) is a fuzzy Baire space, then (X, T) is a fuzzy almost irresolvable space.

Proposition 3.2 : If λ is a fuzzy pre-closed set in a fuzzy perfectly disconnected space (X,T), then int (λ) is a fuzzy regular closed set in (X,T).

Proof : Let λ be a fuzzy pre-closed set in (X,T). Then, 1 − λ is a fuzzy pre- open set in (X,T). Since (X,T) is a fuzzy perfectly disconnected space, by theorem 2.1 , cl (1 − λ) is a fuzzy open set in (X, T) and then , by lemma 2.1, 1 − int (λ) is a fuzzy open set in (X, T) . Hence int (λ) is a fuzzy closed set in (X,T). Then, cl int (λ) = int (λ) in (X,T). Now cl int [int (λ)] = cl int (λ)

= int (λ), implies that int (λ) is a fuzzy regular closed set in (X,T).

space (X, T), then there exists a f

Proposition 3.3 : If λ is a fuzzy residual set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy G_δ –set η in (X,T) such that cl (η) ≤ λ.

Proof : Let λ be a fuzzy residual set in (X,T). Then, by theorem 2.3 , there exists a fuzzy G_δ –set η in (X,T) such that η ≤ λ . Since (X,T)

that cl (η) ≤ int (λ) and then cl (η) ≤ int (λ) ≤ λ. in (X,T). Thus there exists a fuzzy G_δ –set η in (X,T) such that cl (η) ≤ λ.

Proposition 3.4 : If each fuzzy G_δ -set is a fuzzy dense set in a fuzzy perfectly disconnected space (X,T), then 1_X is the only fuzzy residual set in (X,T).

Proof : Let λ be a fuzzy residual set in (X,T). Then, by proposition 3.3, there exists a fuzzy G_δ -set η in (X,T) such that cl (η) ≤ λ. By hypothesis, the fuzzy G_δ - set is a fuzzy dense

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set in (X,T). Then, cl (η) = 1, in (X,T). This implies that 1 ≤ λ . That is, λ = 1, in (X,T). Thus, 1_X is the only fuzzy residual set in (X,T).

Proposition 3.5 : In a fuzzy perfectly disconnected space (X,T), 0_X and 1_X are the only two fuzzy simply open sets in (X,T).

Proof : Let λ be a fuzzy simply open set in (X, T). Then, Bd (λ) is a fuzzy nowhere dense set in (X,T). Since (X,T) is a fuzzy perfectly disconnected space, by theorem 2.4 , Bd(λ) = 0 , in (X,T). That is, [cl (λ) ∧ cl (1 − λ)] = 0, in (X,T). This implies that cl λ) ∧ [1 – int (λ)] = 0 and then cl(λ) ≤ (1 − [1 − int(λ)]) in (X, T). That is, cl (λ) ≤ int (λ). But int (λ) ≤ cl(λ), in (X, T) and hence cl (λ) = int(λ). This is possible if either λ = 0 or 1. Thus, 0_X and 1_X are the only fuzzy simply open sets in a fuzzy perfectly disconnected space.

Proposition 3.6 : If λ is a fuzzy co- σ-boundary set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy G_δ -set δ in (X,T) such that λ ≤ δ .

$i=1$

Proof : Let λ be a fuzzy co- σ-boundary set in (X, T). Then, 1 − λ is a fuzzy σ-boundary set in (X, T). Then, by proposition 3.10 , there exists a fuzzy F_σ -set η in (X,T) such that η ≤ 1− λ . This implies that λ ≤ 1− η . Since η is a fuzzy F_σ -set in (X,T), 1 − η is a fuzzy G_δ –set in (X,T). Let $\delta = 1 - \eta$. Thus, if λ is a fuzzy co- σ-boundary set in (X,T), then there exists a fuzzy G_δ -set δ in (X,T) such that λ ≤ δ .

Proposition 3.7 : If λ is a fuzzy co- σ-boundary set in a fuzzy perfectly disconnected space (X,T) such that cl (λ) = 1, then there exists a fuzzy σ-nowhere dense set η in (X,T) such that η ≤ 1− λ .

Proof : Let λ be a fuzzy co- σ-boundary set in (X, T) such that cl (λ) = 1. Since the topological space (X,T) is fuzzy perfectly disconnected, by proposition 3.11, there exists a fuzzy G_δ -set δ in (X,T) such that λ ≤ δ . This implies that cl (λ) ≤ cl (δ), in (X, T). But cl (λ) = 1, implies that cl (δ) = 1. Thus δ is a fuzzy dense and fuzzy G_δ -set in (X,T). Then, by theorem 2.12, 1− δ is a fuzzy σ-nowhere dense set in (X,T). Let η = 1 − δ . Then η is a fuzzy σ-nowhere dense set in (X,T). Now λ ≤ δ , implies that 1− δ ≤ 1− λ and then η ≤ 1− λ , in (X,T).

Proposition 3.8 : If λ is a fuzzy residual set in a fuzzy second category (but not fuzzy Baire) and fuzzy perfectly disconnected space (X,T), then there exists a fuzzy closed and fuzzy residual set η in (X,T) such that

Proof : Let λ be a fuzzy residual set in a fuzzy second category
(λ), implies that cl [1 − η] ≤ 1 – cl [cl (λ)], in (X,T) and then 1 − int (η) ≤ 1 − cl (λ) . This implies that cl (λ) ≤ int (η), in (X,T).

Proposition 4.1 : If int [$\vee^\infty$ (λ)] = 0, where (λ)'s are fuzzy σ-boundary sets such that int (λ_i) = 0, in a fuzzy perfectly disconnected and fuzzy open hereditarily irresolvable space (X,T) , then (X,T) is a fuzzy Baire space.

Proof : Suppose that int [$\vee^\infty$(λ)] = sets such that int (λ_i) = 0, in a fuzzy perfectly disconnected space (X,T). Then, by proposition 4.2, (X,T) is a fuzzy σ -Baire space. Since (X,T) is a fuzzy open hereditarily irresolvable space, by theorem 2.13, (X, T) is a fuzzy Baire space.

Proposition 4.2 : If cl [$\wedge^\infty(\lambda_i)$] = 1, where(λ_i)'s are fuzzy co- σ-boundary sets such that cl(λ_i) = 1, in a fuzzy perfectly disconnected and fuzzy open hereditarily irresolvable space (X,T) , then (X,T) is a fuzzy Baire space.

Proof : *The proof follows from the proposition 4.3 and theorem 2.13.*

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