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Fixed point theorems in T_0 −quasi-metric spaces

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Abstract-In this paper, we prove unique fixed point theorem for generalized asymptotically non-expansive mappings on q-hyperconvex T_0 −quasi-metric space. Our result unifies, generalize and complement the comparable results from the current literature.

Keyword:-Asymptotically non-expansive mappings; fixed points; q-hyperconvexity; $\boldsymbol{T_0}$ −quasi-metric-space.

1. INTRODUCTION

In 2012 Kunzi and Otafudu [15] have introduced and studied the concept of q-hyperconvex T_0 −quasi-metric space and also present some fixed point theorems for nonexpansive self-maps on a bounded q-hyperconvex quasi- pseudo-metric space. Hyperconvexity was introduced by Aronszajn and Panitchpakdi [1] in 1956. Later, in 1979, independently Sine [19] and Soardi [20] proved that a bounded hyperconvex metric space has a fixed point property for nonexpansive maps and very fundamental properties about hyperconvex spaces as well as the structure of the fixed point set of nonexpansive mappings were shown by Baillon in [2]. Kirk [14] established the fixed point theory for nonexpansive mappings in hyperconvex metric spaces. Khamsi et al. [10] studied fixed points of commuting nonexpansive maps in hyperconvex metric spaces. Since then many interesting works related to a nonexpansive maps have appeared for q-hyperconvex T_0 −quasi- metric spaces, see the reference [3,8,9,15]. In this paper, we continue our studies of this concept by generalizing the above result of Khamsi [13] and shown that an asymptotically nonexpansive mapping on a q-hyperconvex T_0 −quasi-metric space has a unique fixed points.

2. PRELIMINARIES

For the comfort of the peruser and with a specific end goal to settle our wording we review the accompanying ideas.

Definition 2.1: [15] Let Z be a set and let $d: Z \to Z \to [0,\infty)$ be a function mapping into the set $[0,\infty)$ of the nonnegative reals. Then d is called a quasi-pseudometric on Z if

$(a) d(z,z) = 0$ for all $z \in Z$.

$(b)\ d(x,z) \le d(x,y) + d(y,z)\ \forall\, x,y,z \in X$.

We shall say that d is a T_0 −quasi-metric provided that d also satisfies the following condition: For each $x, y \in X$,

$d(x,y) = 0 = d(y,x)$ implies that $x = y$.

Remark 2.2: [13, 15]

- Let d be a quasi-pseudometric on X, then the map $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the conjugate of d.
- It is easy to verify that the function d^s defined by $d^s = d \vee d^{-1}$, i.e.
  $d^s(x,y) = \max\{d(x,y), d(y,x)\}$

Defines a metric on X whenever d is a T_0-quasi-pseudometric.

Definition 2.3.[15] Let (X,d) be a quasi-pseudometric space. For each $x \in X$ and $\epsilon > 0$,

$$B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}.$$

Denotes the open $\epsilon - ball$ at x. The collection of all “open'' balls yields a base for a topology $\tau(d)$. It is called the topology induced by d on X. Similarly, we set for each $x \in X$ and $\epsilon \ge 0$,

$$C_d(x,\epsilon) = \{y \in X: d(x,y) \le \epsilon\}.$$

Denotes the closed ϵ −ball at x.

In some cases we need to replace [0,∞) by [0,∞] (where for a d attaining the value ∞ the triangle inequality is interpreted in the obvious way). In such a case we might discuss a broadened quasi-pseudo

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metric. In the accompanying we now and then apply ideas from the theory of quasi-pseudo metrics to broadened quasi-pseudo metrics (without changing the usual definitions of these concepts).

Definition 2.4. [5] The q-hyperconvex space (X,d) is said to be bicomplete if the metric space (X,d^s) is complete

Example 2.5. Let $X=[0;\infty)$. Define for each $x,y\in X, n(x,y)=x$ if $x>y$, and $n(x,y)=0$ if $x\leq y$. It is not difficult to check that (X,n) is a T_0-quasi-pseudometric space.

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3. Q-HYPERCONVEXITY

In this section, we recall some important results and definitions on q-hyperconvex metric space. Some recent further work about q-hyperconvexity can be found in [9,15,17].

Definition 3.1: [15]A quasi-pseudo-metric space$(X,d)$is called q-hyperconvex provided that for each family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:
if $d(x_i,x_j)\leq r_i+s_j$ whenever $i,j\in I$, then

$$\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset.$$

Remark 3.2: [14] If d and d^{-1} are identical and $r_i=s_i$ for $i\in I$ in definition 3.1, then $(C_d(z_i,r_i))$ and $(C_{d^{-1}}(z_i,s_i))$ coincide and then we recover the well-known definition of hyper convexity due to Aronszajn and Panitchpakdi [1].

Corollary 3.4. [15] Each metric space (Z,m) that is q-hyperconvex (q-hypercomplete) is hyperconvex (hypercomplete).

Corollary 3.5. [17] Each q-hyperconvex T_0 −quasi-metric space (Z,d) is bi-complete.

Example 3.6: [15] Let $\mathbb{R}\times\mathbb{R}$ of the real valued map equipped with the T_0-quasi-metric $d(x,y)=\max\{x-y,0\}$ whenever $x,y\in\mathbb{R}$. Then $(\mathbb{R},d)$ is a q-hyperconvex.

Example 3.7: [14]Consider the product of $(\mathbb{R},u)$ and $(\mathbb{R},u^t)$, that is, $\mathbb{R}^2$ equipped with the q-hyperconvexity $D((\alpha,\beta),(\alpha',\beta'))=(\alpha-\alpha')\vee(\beta'-\beta)$whenever $(\alpha,\beta),(\alpha',\beta')\in\mathbb{R}^2$. Then the diagonal $\{(\alpha,\alpha):\alpha\in\mathbb{R}\}$ in this product q-hyperconvex is isometric to $(\mathbb{R},u^s)$.

Corollary 3.9: [13] The quasi-pseudo-metric subspace $[0,\infty)$ of $(\mathbb{R},u)$ is q-hyperconvex

Definition 3.10. Let (X,d) be a T_0 −quasi-metric space, we say that a mapping $T:X\to X$ has unique fixed points if there exists $0\leq K<\frac{1}{3}$ such that for all $x,y\in X$, the following inquality holds:

$$d(Tx,Ty)\leq K\{d(Tx,x)+d(y,Ty)+d(Tx,y)\}.$$

4. MAIN RESULT

Theorem 4.1. Let (X,d) be a bounded hyperconvex metric space and $T:X\to X$ be asymptotically nonexpansive mapping. Then T has a unique fixed point.

The following result generalizes the above theorem to the setting of q-hyperconvex T_0 −quasi-metric spaces.

Theorem 4.2. Let (X,d) be a bounded q-hyperconvex T_0 −quasi-metric space and let $T:X\to X$ be generalized asymptotically nonexpansive mapping. Then T has unique fixed points.

Proof. Since $T:X\to X$ is generalized asymptotically nonexpansive. Then there exist $0\leq K<\frac{1}{3}$ such that for all $x,y\in X$, then following inequality holds:

$$d(Tx,Ty)\leq K\{d(Tx,x)+d(y,Ty)+d(Tx,y)\}$$

We shall first show that $T:(X,d^s)\to(X,d^s)$ is a generalized asymptotically nonexpansive. Since for any $x,y\in X$, we have

$$d^{-1}(Tx,Ty)=d$$
$$\leq K\{d(Ty,y)+d(x,Tx)+d(y,Tx)\}$$
$$\leq K\{d^{-1}(y,Ty)+d^{-1}(Tx,x)+d^{-1}(Tx,y)\}$$
$$d^{-1}(Tx,Ty)\leq K\{d^{-1}(Tx,x)+d^{-1}(y,Ty)+d^{-1}(Tx,y)\}$$

And we see that $T:(X,d^{-1})\to(X,d^{-1})$ is a generalized asymptotically nonexpansive

Therefore,

$$d(Tx,Ty)\leq K\{d(Tx,x)+d(y,Ty)+d(Tx,y)\}$$
$$\leq K\{d^s(x,Tx)+d^s(y,Ty)+d^s(Tx,y)\}$$

And

$$d^{-1}(Tx,Ty)\leq K\{d^{-1}(y,Ty)+d^{-1}(Tx,x)+d^{-1}(Tx,y)\}$$
$$\leq K\{d^s(x,Tx)+d^s(y,Ty)+d^s(Tx,y)\}$$

For all $x,y\in X$. Hence

$$d^s(Tx,Ty)\leq K\{d^s(x,Tx)+d^s(y,Ty)+d^s(Tx,y)\}$$

For all $x,y\in X$ and so, $T:(X,d^s)\to(X,d^s)$ is a generalized asymptotically nonexpansive.

By assumption,(X,d^s) is bounded q-hyperconvex. Therefore, by Theorem 4.1, T has a unique fixed point.

Corollary 4.3. Let (X,d) be a q-hyperconvex T_0 −quasi-metric space and $T:X\to X$ be a

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nonexpansive mappings. Then T has a unique fixed point.

REFERENCES

[1] Aronszajn, N. & Panitchpakdi, P. (1956): Extension of uniformly continuous transformations and hyperconvexmetric spaces, Pacific J. Math. 6; 405– 439.

[2] Baillon, J.B. (1988): Nonexpansive mappings and hyperconvex spaces, Contemp. Math. 72; 11–19.

[3] Espinola, R. & Khamsi, M. A. (2001): Introduction to hyperconvex spaces, in: Handbook of metric fixed point theory, Kluwer, Dordrecht, pp. 391–435.

[4] Goebel, K. and Kirk, W. A. (1972):A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35(1); pp. 171-174.

[5] Isbell, J. R. (1964): Six theorems about injective metric spaces, Comment. Math. Helvetic, 39; pp. 65–76.

[6] Kabir, Q. A., Jamal, R. and Mohammad, M. (2016): "Review Article" 'A study of Some fixed point theorems and their applications in hyperconvex metric spaces', Imperial Journal of Interdisciplinary Research, 2(12); 1676-1680.

[7] Kabir, Q. A., Mohammad, M., Jamal, R.,& Bhardwaj, R. (2017). Unique fixed points and mappings in hyperconvex metric spaces, International journal of Mathematics and its Applications. 5(3); 171-177.

[8] Kabir, Q. A., Jamal, R., Mohammad, M., & Shabir Q. A., (2017). Structure of $\boldsymbol{u}-$ Injective Hull in Ultra-Quasi-Pseudo Metric Space. International Journal of Scientific and Innovative Mathematical Research, 5(2); pp. 40-46.

[9] Kabir, Q. A., Bhardwaj, R., Jamal, R., & Mohammad, M., (2018). Q-Hyperconvexity in T_0 −Ultra-quasi-metric spaces and existence of fixed point theorems. Aryabhatta journal of Mathematics & Informaticsl, 10(1), 83-88.

[10] Khamsi, M. A., Lin, M. & Sine, R. (1992). On the fixed points of commuting nonexpansive maps in hyperconvex spaces, J. Math. Anal. Appl., 168; 372–380.

[11] Khamsi, M. A., Knaust, H., Nguyen, N. T. & O'Neill, M. D. (2000): Lambda-hyperconvexity in metric spaces, Nonlinear Anal. 43; 21-31

[12] Khamsi, M. A., & Kirk, W. A. (2001): An Introduction to Metric Spaces and Fixed Point Theory, John Wiley, New York,

[13] Khamsi, M. A. (2004): On asymptotically nonexpansive mappings in hyperconvex metric spaces, Proc. Amer. Math. Soc. 132; 365-373.

[14] Kirk, W. A. (1971): A fixed point theorem for mappings with a nonexpansive iterate, Proc. Amer. Math. Soc. 29; 294–298.

[15] Kunzi, H.P., & Otafudu, O. O. (2012): q-Hyperconvexity in quasi-pseudometric spaces and fixed point theorems, J. Function spaces and Applications, 765903,

[16] Kunzi, H.P., & Otafudu, O. O. (2013): The ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space, Appl. Categor. Struct. 21; 651-670.

[17] Mishra, S.N. & Otafudu, O.O. (2013): On asymptotically nonexpansive mappings in q-hyperconvex T_0-quasi-metric- Categorical spaces, Advances in fixed point theory. 3; 135-140.

[18] Salbany, S. (1989): Injective objects and morphisms in Topology and Its Relation to Analysis, Algebra and Combinatorics, World Scientific, Teaneck, NJ, USA,. 394–409.

[19] Sine, R. C. (1989). Hyperconvexity and approximate fixed points, Nonlinear Anal. 13; pp. 863–869.

[20] Soardi, P. (1979): Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73; pp. 25–29.