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Analytical Aspects of Applying Permutation Method in Evaluating a Determinant

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Abstract- The use of permutation method in evaluating a determinant offers several advantages in addition to the determination of the associated value. Among other benefits, the method assists in the sequential generation of the associated terms as well as the permutation cycles of the symmetric group of the same order. Our paper aims to explore the underlying novel concepts of the use of permutation in evaluating a determinant and its application in the modern day mathematical analyses.

Keywords: *permutation, determinant, symmetric group, mathematical analyses*

1. INTRODUCTION

The origin matrices and determinants[1,2] can be traced in the concept that first emerged when resolution of the system of a number of linear equations with the same number of unknowns and were being sought by the mathematical minds back to the second century BC. However, the development really got underway with the reappearance of the idea until the end of 17^{th} century. During evaluation of the determinant of a square matrix $A= (a_{ij})$ of order n[3], it is observed that each term of the determinant consists of the product of the unique element of each row and a unique element of each column. Consequently each term appears as a product of n elements and the value of the determinant as a sum of such n!terms. Since $1 \leq i,j \leq n$ and the corresponding symbol representing the element being a, it is natural that the element in each term can be rearranged (if necessary)such that the value of i(row) runs from 1 to n in the same natural order whilst the n! permutations of the value of j then-after solely account the n! terms of the determinant. Representing the set $\{1,2,3,\ldots\ldots\ldots,n\}$ of the values of i and j by A and the corresponding permutation on A by f, the sign of each term is specified by multiplying it by $(-1)^{k-1}$ where k is the number of elements of A mapped by f to something different. If f partitions A, into disjoint subsets, the rule is applicable separately to each subset; the sign of the term is then the product of the signs (hereafter we specify this rule as the sign-rule). Moreover, from the n! values of j specifying the generation of the terms of the determinant, if we take in account only those values of j for which $f(i) \neq j$, then the values determine the permutation cycles of the symmetric group S_n. We now state and prove the theorem that serves as the foundation of the proposed work.

<u>Theorem1 </u>: The determinant of a square matrix $A=(a_{ij})$ of order n is given by[3],

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{vmatrix}$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{(n-1)!} e_{ij}^{1}\, e_{ij}^{2} e_{ij}^{3} \ldots\ldots\ldots . e_{ij}^{n}$$

where $e_{ij}^{k} \in \{ 1, 2, 3, \ldots\ldots, n\}$, $1\leq k \leq n$, such that $e_{i1}^{1} = 1$ for all I written (n-1)! times vertically downwards as the first column and for $1<k\leq n$, e^{k}_{ij} is one among the remaining (n-(k-1)) symbols each written (n-k)! times following e^{k-1}_{ij} correspondingly, in the respective order 2, 3, …….., n of their occurrence. For j>1 the elements e^{k}_{ij} are obtained by shifting (by adding 1 to the corresponding symbols of the preceding column) and the sign is prefixed as stated in the sign-rule.

Proof: We prove the theorem by induction. To begin with we have for n = 1

$$\sum_{j=1}^{1}\sum_{i=1}^{1} e_{ij}^{1} = e_{11}^{1} = 1 = a_{11}.$$

Hence the theorem holds for n = 1.

For n, $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix} =$
$\sum_{j=1}^{2}\sum_{i=1}^{1!} e_{ij}^{1} e_{ij}^{2} = \sum_{j=1}^{2}(e_{1j}^{1} e_{1j}^{2}) =$
$12 - \sum_{j=2}^{2} e_{1j}^{1} e_{1j}^{2} = 12 - 21 =$
$a11a22 - a12a21$ ....(1)

This shows that the theorem holds for n = 2.

For n = 3, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =\sum_{j=1}^{3}\sum_{i=1}^{2!} e_{ij}^{1} e_{ij}^{2} e_{ij}^{3} =$$

$$\sum_{j=1}^{3} \begin{matrix} e_{1j}^{1} e_{1j}^{2} e_{1j}^{3} \\ e_{2j}^{1} e_{2j}^{2} e_{2j}^{3} \end{matrix} =$$

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$$= \begin{smallmatrix}123\\-132\end{smallmatrix} + \sum_{j=2}^{3} \begin{matrix} e_{1j}^{1}e_{1j}^{2}e_{1j}^{3} \\ e_{2j}^{1}e_{2j}^{2}e_{2j}^{3} \end{matrix} = 123+231+312$$

-132-213-321

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

………………..(2)

This proves that the theorem holds for n = 3 as well.

Now to prove the theorem by induction, let us assume that the theorem holds for a positive integer n = m.

Then,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & & a_{3m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{vmatrix} = \sum_{j=1}^{m}\sum_{i=1}^{(m-1)!} e_{ij}^{1}e_{ij}^{2}e_{ij}^{3} \ldots\ldots\ldots e_{ij}^{m} \ldots\ldots\ldots(3)$$

The signs being as specified in the sign-rule. The expression (3) reveals that for each k, the kth column is the product of k and the corresponding determinant of order m-1 generated from the remaining symbols k+1, k+2, …….m, 1,2, …., k-1 on the basis of the permutation of the digits. This accounts for the (m-1)! terms on each column. Moreover, for each k, the elements of kth column can be obtained by adding k-1 to the corresponding elements of the first column. This means that the validity of the expression for a determinant depends merely on the validity for the determinant of order less by 1 than it. Also the rule of assigning signs holds $\forall$ k $\in$ N. Consequently, the theorem must hold for m+1 assuming it to be true for m. This in other words means that:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(m+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(m+1)} \\ a_{31} & a_{32} & a_{33} & & a_{3(m+1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{(m+1)1} & a_{(m+1)2} & a_{(m+1)3} & & a_{(m+1)(m+1)} \end{vmatrix} = \sum_{j=1}^{m+1}\sum_{i=1}^{m!} e_{ij}^{1}e_{ij}^{2}e_{ij}^{3} \ldots\ldots\ldots e_{ij}^{m+1}$$

Hence by the principle of mathematical induction the theorem holds for all n $\in$ N.

1.1 Formula for determining any specific term

In a determinant of order *n* there are *n*! terms in the form of *n* columns each containing (*n-1*)! terms. Let us assume that we are to determine the m^{th} term where $m \leq n!$. On the basis of the above theorem, since in each column (and hence in the first column) the first element occurs (*n*-1)! times, the second (*n*-2)! times and so forth; the last term must then occur (*n*-*n*)! = 1 time and hence is fixed. Therefore, if we express *m* as a sum of the multiples of (*n*-1)!,(*n*-2)!,…. (*n*-*n*)! taking into consideration that that the multiple of (n-1)!denotes the number of stepsinvolved for the determination of column in which the element lies whilst the multiples of (*n*-2)!, (*n*-3)!,……, (*n*-(*n*-1))! indicate the number of steps involved within the column to determine the position of the 2nd, 3rd,…..(*n*-1)th element respectively. As the last element is the ultimate remaining one or has no choice, the multiple of (*n*-*n*)!being always 1.Thus the task can be accomplished by dividing *m*-1 by (*n*-1)! and then the remainder again by (*n*-2)!and so forth. In the instances where multiple of (*n*-1)! is a non- zero integer we replace it by zero and find the corresponding position in the first column; the non-zero value can then be applied to determine the position of the column. The above discussion leads to the generation of the following proposition.

__**Proposition**__: For any positive integer $m \leq n!$ the m^{th} term of the determinant of A=(a_{ij}) of order *n* the m^{th} term is given by $t_m = \prod_{k=1}^{n} Sn(n-k)!$, where *k* denote the column (of arrangement of elements) and the value of $S_n(0 \leq Sn \leq n-k)$ for *k*=1,2,3,………,*n* denote the steps we move ahead to determine the symbols e^{1}_{ij}, e^{2}_{ij},……..,e^{n}_{ij} respectively constituting the term.

1.2 Formula for determining the distribution of the number of cycles of different lengths

__Proposition:__ In a symmetric group of order *n*(S_n), the number of cycles of length *k*; $k \leq n$ is given by

$$(n-1)(n-2)\ldots\ldots\ldots\ldots.(n-(k-1)) + \cdots\ldots\ldots + (k-1)!/\text{r}!$$

where the first digit in each case denotes the number of columns (in case of 1st term) or groups within the 1st column, starting from the bottom (in case of other terms), between which the cycles with their numbers denoted by the product of the remaining digits in the term are divided. In case of the product of the disjoint cycles we apply the formula successively for each cycle with the subsequent digits and *r* is the number of times the cycle of same length is repeated.

__Proof__: In a determinant of a square matrix *A*=(a_{ij}) of order *n*, there are *n*! elements in the form of *n* columns each containing *(n-1)*!elements. Considering the mapping `*f*` from the set A= {*1,2, 3*,………,*n*} of the values of *i* to the same set *A* of values of *j*, representing the term of the determinant. In this mapping if we take into account only those digits mapped by *f* to something different and rearrange them such that each digit follows its image then this arrangement represents the permutation cycle of the symmetric group S_n. Thus the total number of permutation cycles is *n*!. The permutation cycle is even or odd according to whether the corresponding term is positive or negative. Since a cycle doesn`t

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Vol. 2, No. 4, Dec-2017

change unless the relative position of the elements are changed, considering an element (a digit) as fixed for reference, we just have *n*-1 degree of freedom for the choices out of *n* digits. Thus we have *n*-1 choices for the first and hence *n*-2,*n*-3,*n*-4,………,*n*-(*n-1*),(*n-n)*! choices for the second, thirdfourth,…………,final digit respectively. Further the first 1!, 2!,3!,……..(*n*-1)! terms in the first column represent the permutation cycles of the symmetric group $S_1,S_2, S_3, \ldots, S_{n-1}$respectively. The cycles of similar nature in the last *n*-1 columns have to be added to S_{n-1} so as to account the total cycles of S_n of the same nature. Also the terms that lie in between difference between the consecutive factorial notations like *n*!-(*n*-1)!, ((*n*-1)!-(*n*-2)!), ……,(2!-1!), (1!-0!) indicate the terms where there exist permutations of *n,n-1,n-2,*……..,2 digits respectively. In case of the product of disjoint cycles the number of digits left should be noted carefully while applying for the following. If there are *r* cycles are of same lengths the *r*! interchange among them give rise to no new permutation; so the result is to be divided by *r*!in this case.

2. ILLUSTRATION

Use the permutation method to (a) evaluate the determinant A= (a_{ij}) of order 4 (b) determine it`s 23rd term (c) generate the permutation cycles of S_4

sol^n

(a) We have,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$\sum_{j=1}^{4}\sum_{i=1}^{3!} e^1{}_{ij}\, e^2{}_{ij} e^3{}_{ij} e^4{}_{ij}$$

=				
	1234	– 2341	+ 3412	– 4123
	– 1243	+ 2314	– 3421	+ 4132
	– 1324	– 2431	– 3142	+ 4213
	+ 1342	– 2413	+ 3124	– 4231
	+ 1423	– 2134	+ 3241	– 4312
	– 1432	+ 2143	– 3214	+ 4321

$= a_{11}a_{22}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43}$

$- a_{11}a_{22}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} - a_{13}a_{24}a_{32}a_{41} + a_{14}a_{21}a_{33}a_{42}$

$- a_{11}a_{23}a_{32}a_{44} + a_{12}a_{24}a_{33}a_{41} - a_{13}a_{21}a_{34}a_{42} + a_{14}a_{22}a_{31}a_{43}$

$+ a_{11}a_{23}a_{34}a_{42} - a_{12}a_{24}a_{31}a_{43} + a_{13}a_{21}a_{32}a_{44} - a_{14}a_{22}a_{33}a_{41}$

$+ a_{11}a_{24}a_{32}a_{43} - a_{12}a_{21}a_{33}a_{44} + a_{13}a_{22}a_{34}a_{41} - a_{14}a_{23}a_{31}a_{42}$

$- a_{11}a_{24}a_{33}a_{42} + a_{12}a_{21}a_{34}a_{43} - a_{13}a_{22}a_{31}a_{44} + a_{14}a_{23}a_{32}a_{41}$

(b) We have,

$$23 = 3 \times 6 + 2 \times 2 + 0 \times 1 + 1$$

Replacing 3 by zero, the expression on the right becomes

$$o \times 6 + 2 \times 2 + 0 \times 1 + 1$$

∴ $S_1=0$, $S_2=2$, $S_3=0$, $S_4=1$

∵ $S_1=0$, $e^1{}_{ij}=1$. The digits left in order are 2,3,4. Since $S_2=2$, we move 2 steps forth after 2 which is 4. Therefore, $e^2{}_{ij}=4$. The digits left now in order are 2 and 3. Since $S_3=0$, we don`t move ahead of 2. So $e^3{}_{ij}=2$. The last digit which is then fixed. Therefore, $S_4=3$

So,

The term corresponding to$o \times 6 + 2 \times 2 + 0 \times 1 + 1$ in the first column is 1423. Since, $S_1=3$

shifting each digit 3 steps yields 4312. All the digits are mapped to different symbols; so k=4.Hence $(-1)^{4-1}= -1$. So negative sign is assigned.

∴ The corresponding term is -4312 or $-a_{14}a_{23}a_{31}a_{42}$.

(c)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

=				
	1234	– 2341	+ 3412	– 4123
	– 1243	+ 2314	– 3421	+ 4132
	– 1324	– 2431	– 3142	+ 4213
	+ 1342	– 2413	+ 3124	– 4231
	+ 1423	– 2134	+ 3241	– 4312
	– 1432	+ 2143	– 3214	+ 4321

1234 ↓↓↓↓	1234 ↓↓↓↓	1234 ↓↓↓↓	1234 ↓↓↓↓
1234	– 2341	+ 3412	– 4123
– 1243	+ 2314	– 3421	+ 4132
– 1324	– 2431	– 3142	+ 4213
+ 1342	– 2413	+ 3124	– 4231
+ 1423	– 2134	+ 3241	– 4312
– 1432	+ 2143	– 3214	+ 4321

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Hence the cycles are:

(1)	(1 2 3 4)	(1 3)(2 4)	(1 4 3 2)
(3 4)	(1 2 3)	(1 3 2 4)	(1 4 2)
(2 3)	(1 2 4)	(1 3 4 2)	(1 4 3)
(2 3 4)	(1 2 4 3)	(1 3 2)	(1 4)
(2 4 3)	(1 2)	(1 3 4)	(1 4 2 3)
(2 4)	(1 2)(3 4)	(1 3)	(1 4)(2 3)

(2) Determine the number and the nature of cycles of the symmetric group S_9

Sol^n: In S_9 we consider the permutation on the set A= {1,2,3,……,9}.By the use of formula in afore-mentioned in the proposition, we can easily obtain the number of cycles of different lengths as shown in the table below:
(The explanation of the distribution of cycles is given for the cycles of length 3 in S.N.3)

S.No.	Partition/cycle type	Number of cycles
1	$1+1+1+1+1+1+1+1+1$	1(Identity permutation)
2	$2+1+1+1+1+1+1+1$	(8+7+6+5+4+3+2+1) = 36
3	$3+1+1+1+1+1+1$	$(8\times 7+7\times 6+6\times 5+5\times 4+4\times 3+3\times 2+2\times 1)=168.$ [seven each in the last eight columns + six each in the 7 groups of the first column starting from the bottom each group containing(8!-7!)/7=5040 terms + five each in the six groups of the first column lying exactly on the top of the above 7 groups each containing (7!-6!)/6=720 terms + 4 each in the 5 groups of the 1st column lying exactly on the topof the above6 groups each containing (6!-5!)/5=120 terms + 3 eachin the 4 groups in the 1st column exactly on the top of the above 5 groups each containing (5!-4!)/4=24 terms + 2 each in the 3 groups in the 1st column exactly on the top of the above 4 groups each containing (4!-3!)/3=6 terms + 1eachin the 2 groups in the 1st column exactly on the top of the above 3 groups each containing (3!-2!)/2=2 terms]
4	$4+1+1+1+1+1$	$(8\times 7\times 6+7\times 6\times 5+6\times 5\times 4+5\times 4\times 3+4\times 3\times 2+3\times 2\times 1)=756$
5	$5+1+1+1+1$	$8\times 7\times 6\times 5+7\times 6\times 5\times 4+\cdots\cdots\cdots+4!=3024$
6	$6+1+1+1$	$8\times 7\times 6\times 5\times 4+\cdots\cdots\cdots\cdots+5!=10080$
7	$7+1+1$	$8\times 7\times 6\times 5\times 4\times 3+\cdots\cdots\cdots+6!=25920$
8	$8+1$	$8\times 7\times 6\times 5\times 4\times 3\times 2+7!=45360$
9	9	$8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=8!=40320$
10	$2+2+1+1+1+1+1$	(Number of cycles of length 2 from 9 digits)(Number of cycles of length 2 from the remaining 7 digits)/2!=$36(6+5+4+3+2+1)/2!=\frac{36\times 21}{2}=378$
11	$2+2+2+1+1+1$	(Number of cycles of length 2 from 9 digits)(Number of cycles of length 2 from the remaining 7 digits)(Number of cycles of length2 from the last remaining 5 digits)/3!=$(36\times 21\times 10)/3!=1260$
12	$3+3+1+1+1$	$(168(5\times 4+4\times 3+3\times 2+2\times 1)/2!=3360$
13	$2+3+1+1+1+1$	$36(6\times 5+5\times 4+4\times 3+3\times 2+2\times 1)=168(5+4+3+2+1)=2520$
14	$2+4+1+1+1$	$36(6\times 5\times 4+5\times 4\times 3+4\times 3\times 2+3\times 2\times 1)=756(4+3+2+1)=7560$
15	$2+2+3+1+1$	$36(6+5+4+3+2+1)(4\times 3+\cdots\cdots+2!)=36\times 21\times 20=7560$
16	$4+3+1+1$	$756(4\times 3+\cdots\cdots+2!)=15120$
17	$2+5+1+1$	$36(6\times 5\times 4\times 3+\cdots\cdots\cdots 4!)=3024(3+2+1)=18144$
18	$2+2+2+2+1$	$36\times 21\times 10\times 3/4!=945$
19	$3+3+2+1$	$168\times 40\times 3/2!=36\times 70\times 8/2!=10080$
20	$4+4+1$	$756\times 30/2!=11340$
21	$5+3+1$	$3024\times 8=168\times 144=24192$

22	$6+2+1$	$10080 \times 3 = 30240$
23	$2+2+4+1$	$36 \times 21 \times 30/2! = 11340$
24	$2+2+2+3$	$36 \times 21 \times 10 \times 2/3! = 2520$
25	$2+2+5$	$36 \times 21 \times 24/2! = 9072$
26	$2+7$	$36 \times 720 = 25920$
27	$3+3+3$	$168 \times 40 \times 2!/3! = 2240$
28	$3+6$	$168 \times 5! = 20160$
29	$2+3+4$	$36 \times 70 \times 3! = 15120$
30	$4+5$	$756 \times 24 = 18144$

Conclusion

The use of the permutation method while evaluating a determinant of order n reveals the sequential generation of it`s $n!$ termsin the form of n columns each containing $(n-1)!$ terms. The concept of shifting helps in the instant generation of the elements of a column from the corresponding elements of the first column and specification of any particular term. The process involved herein helps to develop the idea generation of the permutation cycles of $S_n$of varying lengths and their sequential distribution. Moreover, the concept can be applied to find out the distribution of the cycles of S_{n-1} of similar type from that of S_n.

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