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Modified maximum likelihood estimation of parameters in the log-logistic distribution under progressive Type II censored data with binomial removals

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Abstract- In this paper, the estimation problem for the unknown parameters of the log- logistic distribution based on progressive type-II censored data with random removals is considered, where the number of units removed at each failure time follows a binomial distribution. We derive the modified maximum likelihood estimators using the approach of Tiku and Suresh [16] and Suresh [14], as the likelihood equations are intractable of the unknown parameters. Also their asymptotic variance-covariance matrix of the estimates is obtained. Further, reliability characteristics of distribution, confidence intervals and coverage probabilities of estimators are studied.

Index Terms- log- logistic distribution, binomial removals, maximum likelihood estimation, modify maximum likelihood estimation, progressive type-II censoring, variance- covariance matrix, reliability characteristics, coverage probabilities

1. INTRODUCTION

In probability and statistic the log-logistic distribution, which is also known as Fisk distribution in economics is a continuous probability distribution for a non-negative random variables. It has gained special attention due to the importance of using it in practical situation. It is used in survival analysis as a parametric model for events whose rate increases initially and decreases later, for example mortality from cancer following diagnosis or treatment. It has also been used in hydrology to model stream flow and precipitation (Rowinski et al. [8]), and in economics as a simple model of the distribution of wealth or income. It is also used to concentration-response studies (Straetemans et al. [13]) in oncology.

Censored sampling arises in a life testing experiment whenever the experimenter does not observe the failure times of all items placed on a life test. There are many cases in life testing experiments in which units are lost or removed from the test before failure. In medical or industrial applications, researchers have to treat the censored data because they usually do not have sufficient time to observe the lifetime of all subjects in the study.

A type II censored sample is one for which m smallest observations in a sample of n items are observed. A generalization of type II censoring is a progressive type II censoring. Under this scheme, n units of the same kind are placed on test at time zero, and m failures are observed. When the first failure is observed, a number r_1 of surviving units are randomly withdrawn from the test; at the second failure time, r_2 surviving units are selected at random and taken out of the experiment, and so on. At the time of the m^{th} failure, the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ units are removed. Progressive censoring is useful in both industrial life testing applications and clinical settings; it allows the removal of surviving experimental units before the termination of the test. Balakrishnan and Aggarwala [2] provided a comprehensive reference on the subject of progressive censoring and its applications. Balkrishnan et al. [3] indicated that such scheme can arise in clinical trials where the drop out of patients may be caused by migration or by lack of interest. In some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the pattern of removal at each failure is random. We assume that, any test unit being dropped out from the life test is independent of the others but with the same probability p. In such situation, the progressive censoring scheme with random removals is required. For more information one can also refer Balkrishnan and Aggrawala [2]. If $r_i = 0\,; i = 1,2,\dots,m-1$ then progressive censoring reduces to Type II censoring and if $r_i = 0\,; i = 1,2,\dots,m$ and $m = n$ then the scheme reduces to no censoring i.e. case of complete observed sample. Note that, in this scheme, $R_1, R_2, \dots, R_m$ are all pre-fixed.

Some related works can be found, for example, Tse et al. [17] derived the maximum likelihood estimators of the parameters of Weibull distribution and their asymptotic variance-covariance based on progressive type-II censoring with binomial removals. Wu and

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Chang [18] discussed the estimation problem based on exponential progressive type-II censored data with binomial removals. Wu [19] obtained the estimators of the parameters for Pareto distribution based on progressive type-II censoring with uniform removals. Shuo and Chun [10] considered the estimation problem for the Pareto distribution based on progressive Type II censoring with random removals. Mubarak [7] discussed maximum likelihood estimation for parameter estimation based on the Frechet Progressive Type II Censored data with binomial removals. Shanubhogue and Jain [9] studied minimum variance unbiased estimation in Pareto distribution of first kind under progressive Type II censored data with binomial removals. Srinivasa Rao et al. [12] studied reliability estimation problems for the log-logistic distribution from censored samples using modified maximum likelihood approach. Further, Srinivasa Rao [11] studied estimation of system reliability for log-logistic distribution.

In this paper we discuss the problem of estimation of location parameters μ and scale parameter β of the log-logistic distribution based on progressive type II censored data with binomial removals. The parameters in log-logistic distribution are estimated by classical method of estimation namely maximum likelihood estimation (MLE), and it is an iterative solution of ML equations. Approximations and modifications to the maximum likelihood (ML) method of estimation in certain distributions to overcome iterative solutions of ML equations for the parameters suggested by many authors [for examples Tiku and Suresh [16], Suresh [14], Asgharzedsh [1] etc]. Tiku and Suresh [16] modify maximum likelihood estimators as the likelihood equations are intractable of the unknown parameters obtained modified maximum likelihood (MML) estimates by making linear approximations. Further Srinivasa Rao G. et al. [12] obtain MML estimate of scale parameter in two parameter log-logistic distribution. This paper is organized as follows. In Section 2, we give model assumption and derive the likelihood of log-logistic distribution under progressive type II censored data with binomial removals. Section 3 deals with estimation of parameters by maximum likelihood estimation (MLE). In Section 4 we discuss the modified maximum likelihood estimation (MMLE). In Section 5, we discuss the observed Fisher information matrix, reliability characteristics, confidence intervals and coverage probabilities of estimators.

2. MODEL ASSUMPTION AND LIKELIHOOD FUNCTION

Suppose that the lifetime X of the unit follows the log-logistic distribution with parameters α, μ and β. The probability density function, cumulative distribution function, reliability function and hazard rate function of the log- logistic distribution are given respectively by:

$$f(x) = \frac{\frac{\alpha}{\beta}\left(\frac{x-\mu}{\beta}\right)^{\alpha-1}}{\left[1+\left(\frac{x-\mu}{\beta}\right)^{\alpha}\right]^2} \quad , x > \mu, \alpha > 0, \beta > 0 \quad (2.1)$$

$$F(x) = \frac{\left(\frac{x-\mu}{\beta}\right)^{\alpha}}{1+\left(\frac{x-\mu}{\beta}\right)^{\alpha}} , x > \mu, \alpha > 0, \beta > 0 \quad (2.2)$$

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{\alpha}{\beta}\left(\frac{x-\mu}{\beta}\right)^{\alpha-1}}{1+\left(\frac{x-\mu}{\beta}\right)^{\alpha}}, x > \mu, \alpha > 0, \beta > 0 \quad (2.3)$$

where the parameters α , μ and β are shape parameter, location parameter and scale parameter respectively of the distribution.

Let $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$, denote the progressively Type II censored sample where $X_1 < X_2 < \cdots < X_m$

Using the above mentioned assumptions, the conditional likelihood function for type-II progressively censored model defined by Cohen [4] as follows:

$$L(\alpha, \mu, \beta; \underline{x}/\underline{R} = \underline{r}) = C \prod_{i=1}^{m} f(x_i; \alpha, \mu, \beta)\left(1 - F(x_i)\right)^{r_i} \quad (2.4)$$

Where $C = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - \sum_{i=1}^{m-1} r_i - m + 1)$ and r_i can be any integer between 0 and $(n - m - r_1 - r_2 - \cdots - r_{i-1})$ for $i = 1,2, \dots, m - 1$. Here $r_0 = 0$.

Thus the constant C is the number of ways in which the m progressively type-II censored ordered statistics may select.

The conditional likelihood function of the log-logistic distribution is

$$L(\alpha, \mu, \beta; \underline{x}/\underline{R} = \underline{r}) = C \prod_{i=1}^{m} \left\{\frac{\frac{\alpha}{\beta}\left(\frac{x_i-\mu}{\beta}\right)^{\alpha-1}}{\left[1+\left(\frac{x_i-\mu}{\beta}\right)^{\alpha}\right]^2}\right\}\left\{\frac{1}{1+\left(\frac{x_{(i)}-\mu}{\beta}\right)^{\alpha}}\right\}^{r_i}$$

$$= C\frac{\alpha^m}{\beta^m} \prod_{i=1}^{m} \left\{\frac{\left(\frac{x_i-\mu}{\beta}\right)^{(\alpha-1)}}{\left[1+\left(\frac{x_i-\mu}{\beta}\right)^{\alpha}\right]^{2+r_i}}\right\} \quad (2.5)$$

Suppose that the numbers of removed items are independent and have the identical probability mass function

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-r)^{n-m-r_1} \quad ; \quad r_1 = 0,1,2, \dots, n - m \quad (2.6)$$

and

$$P(R_i = r_i/R_{i-1} = r_{i-1}, R_{i-2} = r_{i-2}, \dots, R_1 = r_1) = \binom{n - m - \sum_{l=1}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=1}^{i} r_l} \quad (2.7)$$

Where, $r_i = 0,1,2, \dots, n - m - (r_1 + r_2 + \cdots + r_{i-1})$, $i = 1,2, \dots, m - 1$

Suppose that R_i is independent of x_i, so we can write

$$P(\underline{R}, p) = P(R_{m-1} = r_{m-1}/R_{m-2} = r_{m-2}, \cdots, R_1 = r_1)P(R_{m-2} = r_{m-2}/R_{m-3} = r_{m-3}, \cdots, R_1 = r_1) * \cdots * P(R_2 = r_2/R_1 = r_1) * P(R_1 = r_1)$$

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Therefore,

$$P(R,p)=\frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1}r_i)!\prod_{i=1}^{m-1}r_i!}p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i} \quad (2.8)$$

The likelihood function of progressive type-II censoring with random removals is defined as follows, using the assumptions that r_i is independent of x_i for all i:

$$L(\alpha,\mu,\beta,p,\underline{x},\underline{R})=L(\alpha,\mu,\beta,\underline{x}/\underline{R}=\underline{r})P(\underline{R}=\underline{r}) \quad (2.9)$$

where $L(\alpha,\mu,\beta,\underline{x}/\underline{R}=\underline{r})$ is the likelihood function for a progressive type II censored scheme defined in (2.5).

Let $z_i=\frac{x_i-\mu}{\beta}$, $\quad i=1,2,\cdots,m$

The likelihood function (2.9) is

$$L(\alpha,\mu,\beta,p,\underline{x},\underline{R})=\frac{C(n-m)!}{(n-m-\sum_{i=1}^{m-1}r_i)!\prod_{i=1}^{m-1}r_i!}\left(\frac{\alpha}{\beta}\right)^m\prod_{i=1}^{m}\frac{z_i^{(\alpha-1)}}{(1+z_i^{\alpha})^{2+r_i}}p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i} \quad (2.10)$$

$$L(\alpha,\mu,\beta,p,\underline{x},\underline{R})=C^*\left(\frac{\alpha}{\beta}\right)^m\prod_{i=1}^{m}\frac{z_i^{(\alpha-1)}}{(1+z_i^{\alpha})^{2+r_i}}p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i} \quad (2.11)$$

where $C^*=\frac{C(n-m)!}{(n-m-\sum_{i=1}^{m-1}r_i)!\prod_{i=1}^{m-1}r_i!}$

3. ML ESTIMATION OF PARAMETERS $\alpha,\mu,\ \beta$ AND p

The log likelihood function of equation (2.11) is given by

$$l=lnL(\alpha,\mu,\beta,p,\underline{x},\underline{R})=\ln(C^*)+mln\alpha-mln\beta+(\alpha-1)\sum_{i=1}^{m}lnz_i-\sum_{i=1}^{m}(2+r_i)\ln(z_i{}^{\alpha}+1)+\sum_{i=1}^{m-1}r_ilnp+((n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i)\ln(1-p) \quad (3.1)$$

Since the probability $P(\underline{R}=\underline{r})$ is free from parameters α,μ and β, the estimation of this parameter can be directly driven from log of equation (2.8)

$$\frac{\partial lnP(\underline{R}=\underline{r})}{\partial p}=\frac{\sum_{i=1}^{m-1}r_i}{p}-\frac{[(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i]}{1-p}=0$$

$$(1-p)\sum_{i=1}^{m-1}r_i-(n-m)(m-1)p+p\sum_{i=1}^{m-1}(m-i)r_i=0$$

$$\sum_{i=1}^{m-1}r_i-p[\sum_{i=1}^{m-1}r_i+(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i]=0$$

Therefore, we have

$$\hat{p}=\frac{\sum_{i=1}^{m-1}r_i}{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i-1)r_i} \quad (3.2)$$

Differentiate (3.1) with respect to α, μ and β, we obtain

$$\frac{\partial l}{\partial\alpha}=\frac{m}{\alpha}+\sum_{i=1}^{m}lnz_i-\sum_{i=1}^{m}(2+r_i)\left(\frac{z_i^{\alpha}lnz_i}{1+z_i^{\alpha}}\right)=0 \quad (3.3)$$

$$\frac{\partial l}{\partial\mu}=(\alpha-1)\sum_{i=1}^{m}\frac{1}{z_i}\left(\frac{-1}{\beta}\right)-\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i{}^{\alpha})}\alpha z_i^{\alpha-1}\left(\frac{-1}{\beta}\right)=0$$

Therefore,

$$\frac{\partial l}{\partial\mu}=-\left(\frac{\alpha-1}{\beta}\right)\sum_{i=1}^{m}z_i{}^{-1}+\left(\frac{\alpha}{\beta}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i{}^{\alpha})}z_i^{\alpha-1}=0 \quad (3.4)$$

$$\frac{\partial l}{\partial\mu}=\left(\frac{\alpha-1}{\beta}\right)\sum_{i=1}^{m}z_i{}^{-1}-\left(\frac{\alpha}{\beta}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i{}^{\alpha})}z_i^{\alpha-1}=0$$

$$\frac{\partial l}{\partial\beta}=-\frac{m}{\beta}+(\alpha-1)\sum_{i=1}^{m}\frac{1}{z_i}\left(\frac{-z_i}{\beta}\right)-\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i{}^{\alpha})}\alpha z_i^{\alpha-1}\left(\frac{-z_i}{\beta}\right)=0$$

Therefore,

$$\frac{\partial l}{\partial\beta}=-\frac{m}{\beta}-\left(\frac{\alpha-1}{\beta}\right)\sum_{i=1}^{m}z_i{}^{-1}z_i+\left(\frac{\alpha}{\beta}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_{i)}{}^{\alpha})}z_i^{\alpha}=0 \quad (3.5)$$

$$\frac{\partial l}{\partial\beta}=\frac{m}{\beta}+\left(\frac{\alpha-1}{\beta}\right)\sum_{i=1}^{m}z_i{}^{-1}z_i-\left(\frac{\alpha}{\beta}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_{i)}{}^{\alpha})}z_i^{\alpha}=0$$

.

Equations should be referred to in abbreviated form, e.g. “Eq. (1)”. In multiple-line equations, the number should be given on the last line.

Displayed equations are to be centered on the page width. Standard English letters like x are to appear as *x* (italicized) in the text if they are used as mathematical symbols. Punctuation marks are used at the end of equations as if they appeared directly in the text.

4. MODIFIED MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS μ AND β

We assume here that shape parameter α in equation (2.1) is known. The ML equations (3.4) and (3.5) do not have explicit solution for μ and β. This is due to the fact that the term $g(z)=\frac{Z^{\alpha-1}}{1+Z^{\alpha}}$ is intractable. The solution of equations (3.4) and (3.5) can be evaluated numerically by some suitable interative procedure such a Newton-Rapshon method for given values of $(n,m,\underline{R},\underline{x})$. To derive the modified maximum likelihood estimators which are asymptotically fully efficient we will use Tiku and Suresh [16] approach to approximate ML equations for μ and β by linearizing the term $g(z)=\frac{Z^{\alpha-1}}{1+Z^{\alpha}}$ using the Taylor series expansion around the quantile point of F i.e. $t_{(i)}=E(Z_{(i)})$.

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First order of approximation of ${z_i}^{-1}$ and $g(z_i) = \frac{{z_i}^{\alpha-1}}{1+{z_i}^{\alpha}}$ around quantile point of F are:

$${z_i}^{-1} = \alpha_{1_i} - \beta_{1_i} z_i \quad (4.1)$$

where

$\beta_{1_i} = -\left[\frac{d}{dz} z^{-1} = -z^{-2}\right]_{z=t_i} = {t_i}^{-2}$; $\alpha_{1_i} = g_1(t_i) + \beta_{1_i} g_1{}'(t_i) = {t_i}^{-1} + {t_i}^{-2} t_i = 2{t_i}^{-1}$

$$g_2(z_i) = g_2(t_i) + (z_i - t_i) g_2{}'(t_i)$$
$$= (g_2(t_i) - t_i g_2{}'(t_i)) + g_2{}'(t_i) z_i$$
$$= \alpha_{2_i} + \beta_{2_i} z_i \quad (4.2)$$

Where

$$\beta_{2_i} = g_2{}'(t_i)$$
$$= \left[\frac{d}{dz}\frac{z^{\alpha-1}}{1+z^{\alpha}} = \frac{(1+z^{\alpha})(\alpha-1)z^{\alpha-2} - z^{\alpha-1}\alpha z^{\alpha-1}}{(1+z^{\alpha})^2} = \frac{(\alpha-1)z^{\alpha-2} - z^{2\alpha-2}}{(1+z^{\alpha})^2}\right]_{z=t_i}$$

$$= \frac{(\alpha-1){t_i}^{\alpha-2} - {t_i}^{2\alpha-2}}{(1+{t_i}^{\alpha})^2}$$ and

$$\alpha_{2_i} = g(t_i) - t_i g'(t_i)$$
$$= \frac{{t_i}^{\alpha-1}}{(1+{t_i}^{\alpha})} - t_i * \left\{\frac{(\alpha-1){t_i}^{\alpha-2} - {t_i}^{2\alpha-2}}{(1+{t_i}^{\alpha})^2}\right\}$$

$$= \frac{(1+{t_i}^{\alpha}){t_i}^{\alpha-1} - (\alpha-1){t_i}^{\alpha-1} + {t_i}^{2\alpha-1}}{(1+{t_i}^{\alpha})^2} = \frac{(2-\alpha){t_i}^{\alpha-1} + 2{t_i}^{2\alpha-1}}{(1+{t_i}^{\alpha})^2}$$

The approximate value of t_i obtained from the solving equation

$$\int_0^{t_i} f(z)\,dz = \frac{i}{n+1} = q_i$$
$$\alpha \int_0^{t_i} \frac{z^{\alpha-1}}{(1+z^{\alpha})^2}\,dz = \frac{i}{n+1} = q_i$$

We have $t_i = \left(\frac{q_i}{1-q_i}\right)^{1/\alpha}$ for $i = 1,2,\dots,m$

Substitute the values of equation (4.1) and (4.2) in equations (3.4) and (3.5), we have the modified likelihood equations are

$$\frac{\partial l^*}{\partial \mu} \cong \frac{\partial l}{\partial \mu} = -\left(\frac{\alpha-1}{\beta}\right)\sum_{i=1}^{m}(\alpha_{1_i} - \beta_{1_i} z_i) + \frac{\alpha}{\beta}\sum_{i=1}^{m}(2+r_i)(\alpha_{2_i} + \beta_{2_i} z_i) = 0$$

$$-\frac{1}{\beta}\sum_{i=1}^{m}\left[(\alpha-1)\alpha_{1_i} - \alpha(2+r_i)\alpha_{2_i}\right] + \frac{1}{\beta}\sum_{i=1}^{m}\left[(\alpha-1)\beta_{1_i} + \alpha(2+r_i)\beta_{2_i}\right] z_i = 0$$

$$\frac{-1}{\beta}\sum_{i=1}^{m}\Delta_i + \frac{1}{\beta}\sum_{i=1}^{m}\delta_i z_i = 0 \quad (4.3)$$

Where $\Delta_i = \left[(\alpha-1)\alpha_{1_i} - \alpha(2+r_i)\alpha_{2_i}\right]$ and

$$\delta_i = \left[(\alpha-1)\beta_{1_i} + \alpha(2+r_i)\beta_{2_i}\right]$$

Therefore,

$$\frac{-1}{\beta}\sum_{i=1}^{m}\Delta_i + \frac{1}{\beta}\sum_{i=1}^{m}\delta_i\left(\frac{x_i-\mu}{\beta}\right) = 0$$
$$\frac{1}{\beta}\sum_{i=1}^{m}\Delta_i - \frac{1}{\beta^2}\sum_{i=1}^{m}\delta_i x_i + \frac{1}{\beta^2}\sum_{i=1}^{m}\delta_i \mu = 0$$
$$\beta\sum_{i=1}^{m}\Delta_i - \sum_{i=1}^{m}\delta_i x_i + \mu Q = 0$$
$$\widehat{\beta^*}\frac{\sum_{i=1}^{m}\Delta_i}{Q} - \frac{\sum_{i=1}^{m}\delta_i x_i}{Q} + \widehat{\mu^*} = 0$$

Hence,

$$\widehat{\mu^*} = K - \Delta\widehat{\beta^*} \quad (4.4)$$

where $Q = \sum_{i=1}^{m}\delta_i$, $\Delta = \frac{\sum_{i=1}^{m}\Delta_i}{Q}$ and $K = \frac{\sum_{i=1}^{m}\delta_i x_i}{Q}$

$$\frac{\partial l^*}{\partial \beta} \cong \frac{\partial l}{\partial \beta} = -\frac{m}{\beta} - \frac{\alpha-1}{\beta}\sum_{i=1}^{m}(\alpha_{1_i} - \beta_{1_i} z_i) z_i + \frac{\alpha}{\beta}\sum_{i=1}^{m}(2+r_i)(\alpha_{2_i} + \beta_{2_i} z_i) z_i = 0$$

$$\Rightarrow -\frac{m}{\beta} - \frac{1}{\beta}\sum_{i=1}^{m}\left[(\alpha-1)\alpha_{1_i} - \alpha(2+r_i)\alpha_{2_i}\right] z_i + \frac{1}{\beta}\sum_{i=1}^{m}\left[(\alpha-1)\beta_{1_i} + \alpha(2+r_i)\beta_{2_i}\right] {z_i}^2 = 0$$

$$\Rightarrow -\frac{m}{\beta} - \frac{1}{\beta}\sum_{i=1}^{m}\Delta_i\left(\frac{x_i-\mu}{\beta}\right) + \frac{1}{\beta}\sum_{i=1}^{m}\delta_i\left(\frac{x_i-\mu}{\beta}\right)^2 = 0$$

$$\Rightarrow \frac{m}{\beta} + \frac{1}{\beta}\sum_{i=1}^{m}\Delta_i\left(\frac{x_i-K+K+\mu}{\beta}\right) - \frac{1}{\beta}\sum_{i=1}^{m}\delta_i\left(\frac{x_i-K+K-\mu}{\beta}\right)^2 = 0$$

$$\Rightarrow \frac{m}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{m}\Delta_i(x_i-K) + \frac{(K-\mu)}{\beta^2}\sum_{i=1}^{m}\Delta_i - \frac{1}{\beta^3}\sum_{i=1}^{m}\delta_i(x_i-K)^2 - \frac{(K-\mu)^2}{\beta^3}\sum_{i=1}^{m}\delta_i - \frac{2(K-\mu)}{\beta^3}\sum_{i=1}^{m}\delta_i(x_i-K) = 0$$

Since, $\sum_{i=1}^{m}\delta_i(x_i-K) = 0$ and using equation (4.4)

We have,

$$\frac{m}{\beta} + \frac{B}{\beta^2} - \frac{C}{\beta^3} = 0 \Rightarrow m\beta^2 + B\beta - C = 0 \quad (4.5)$$

Therefore we have,

$$\widehat{\beta^*} = \frac{-B \pm \sqrt{B^2+4mC}}{2m} \quad (4.6)$$

where $B = \sum_{i=1}^{m}\Delta_i(x_i-K)$

and $C = \sum_{i=1}^{m}\delta_i(x_i-K)^2$

It should be mentioned here that upon solving equation (4.5) for β, we obtain a quadratic equation in β which has two roots; however one of them dropouts since $\beta > 0$. For skewed distribution $\Delta = 0$, off-course the divisor $2m$ may be replaced by $2\sqrt{m(m-1)}$ to reduce the bias if any. If $\delta_i > 0$ for

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any $i = 1,2,\ldots,m$ the $\hat{\beta}$ is real and positive. (Tiku ad Akkaya, p.29). Therefore,

$$\widehat{\beta^*} = \frac{-B+\sqrt{B^2+4mC}}{2\sqrt{m(m-1)}} \qquad (4.7)$$

The appropriate MMLES in (4.4) and (4.7) may provide as with a good starting value for the iterative solution of the likelihood equations (3.4) and (3.5).

5. OBSERVED FISHER INFORMATION, RELIABILITY CHARACTERISTICS, CONFIDENCE INTERVALS AND COVERAGE PROBABILITIES

Using the log-likelihood equations in (3.4) and (3.5), we obtain the observed Fisher information as

$$\frac{\partial^2 l}{\partial\mu^2} = -\frac{\alpha-1}{\beta}\sum_{i=1}^{m}\left(-\frac{1}{z_i^2}\right) + \frac{\alpha}{\beta}\sum_{i=1}^{m}(2 + r_i)\left\{\frac{(1+z_i^{\alpha})(\alpha-1)z_i^{\alpha-2}\left(\frac{-1}{\beta}\right) - z_i^{\alpha-1}\alpha z_i^{\alpha-1}\left(\frac{-1}{\beta}\right)}{(1+z_i^{\alpha})^2}\right\}$$

$$\frac{\partial^2 l}{\partial\mu^2} = \left(\frac{\alpha-1}{\beta^2}\right)\sum_{i=1}^{m}(z_i)^{-2} - \left(\frac{\alpha}{\beta^2}\right)\sum_{i=1}^{m}(2 + r_i)\left\{\frac{(1+z_i^{\alpha})(\alpha-1)z_i^{\alpha-2} - \alpha z_i^{2\alpha-2}}{(1+z_i^{\alpha})^2}\right\} \qquad (5.1)$$

$$\frac{\partial^2 l}{\partial\beta^2} = \frac{m}{\beta^2} + m\left(\frac{\alpha-1}{\beta^2}\right) - \left(\frac{\alpha^2}{\beta^2}\right)\sum_{i=1}^{m}(2 + r_i)\frac{z_i^{\alpha}}{(1+z_i^{\alpha})^2} \qquad (5.2)$$

$$\frac{\partial^2 l}{\partial\mu\partial\beta} = -\frac{\alpha-1}{\beta}\sum_{i=1}^{m}(-1)(z_i)^{-2}\left(\frac{-z_i}{\beta}\right) - (\alpha - 1)\sum_{i=1}^{m} z_i^{-1}\left(-\frac{1}{\beta^2}\right) + \frac{\alpha}{\beta}\sum_{i=1}^{m}(2 + \gamma_i)\left\{\frac{(1+z_i^{\alpha})(\alpha-1)Z_i^{\alpha-2}\left(-\frac{z_i}{\beta}\right) - Z_i^{\alpha-1}\alpha Z_i^{\alpha-1}\left(\frac{-z_i}{\beta}\right)}{(1+z_i^{\alpha})^2}\right\} + \left(\frac{-\alpha}{\beta^2}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i^{\alpha})} z_i^{\alpha-1}$$

$$= -\frac{\alpha^2}{\beta^2}\sum_{i=1}^{m}(2 + \gamma_i)\left\{\frac{(1+z_i^{\alpha})(\alpha-1)Z_i^{\alpha-1} - Z_i^{2\alpha-1}\alpha Z_i^{\alpha-1}}{(1+z_i^{\alpha})^2} + \frac{z_i^{\alpha-1}}{(1+z_i^{\alpha})}\right\}$$

$$\frac{\partial^2 l}{\partial\mu\partial\beta} = -\left(\frac{\alpha^2}{\beta^2}\right)\sum_{i=1}^{m}\frac{(2+r_i)}{(1+z_i^{\alpha})^2} z_i^{\alpha-1} \qquad (5.3)$$

We derive the similar expansions for the appropriate likelihood equations (4.3) and (4.5):

$$\frac{\partial^2 l^*}{\partial\mu^2} \cong \left(\frac{-1}{\beta^2}\right)\sum_{i=1}^{m}\delta_i \qquad (5.4)$$

$$\frac{\partial^2 l^*}{\partial\beta^2} \cong \frac{m}{\beta^2} - \left\{\left(\frac{1}{\beta}\right)\sum_{i=1}^{m}\Delta_i\left(\frac{-Z_i}{\beta}\right) + \sum_{i=1}^{m}(\Delta_i z_i)\left(\frac{-1}{\beta^2}\right)\right\} + \left\{\left(\frac{1}{\beta}\right)\sum_{i=1}^{m}(\delta_i 2z_i)\left(\frac{-Z_i}{\beta}\right) + \sum_{i=1}^{m}(\delta_i z_i^2)\left(\frac{-1}{\beta^2}\right)\right\}$$

$$\cong \frac{m}{\beta^2} + \left(\frac{2}{\beta^2}\right)\sum_{i=1}^{m}(\Delta_i z_i) - \left(\frac{3}{\beta^2}\right)\sum_{i=1}^{m}(\delta_i z_i^2) \qquad (5.5)$$

$$\frac{\partial^2 l^*}{\partial\mu\partial\beta} \cong \left(\frac{\sum_{i=1}^{m}\Delta_i}{\beta^2}\right) + \left[\left(\frac{1}{\beta}\right)\sum_{i=1}^{m}\delta_i\left(\frac{-Z_i}{\beta}\right) + \sum_{i=1}^{m}(\delta_i z_i)\left(\frac{-1}{\beta^2}\right)\right] \cong \left(\frac{\sum_{i=1}^{m}\Delta_i}{\beta^2}\right) - \left(\frac{2}{\beta^2}\right)\sum_{i=1}^{m}(\delta_i z_i) \qquad (5.6)$$

From these expressions of observed Fisher information, we can obtain asymptotic variance of the approximate MLE (instead of basing it on the expected Fisher information). Unfortunately, the exact mathematical expression for the expected Fisher information is difficult to obtain.

$$E\left(-\frac{\partial^2 l}{\partial\mu^2}\right) = \left(\frac{\alpha-1}{\beta^2}\right)\sum_{i=1}^{m} E(-(z_i)^{-2}) + \left(\frac{\alpha}{\beta^2}\right)\sum_{i=1}^{m}(2 + r_i)E\left\{\frac{(1+z_i^{\alpha})(\alpha-1)z_i^{\alpha-2} - \alpha z_i^{2\alpha-2}}{(1+z_i^{\alpha})^2}\right\}$$

$$E\left(-\frac{\partial^2 l}{\partial\beta^2}\right) = -\frac{m}{\beta^2} - m\left(\frac{\alpha-1}{\beta^2}\right) + \left(\frac{\alpha^2}{\beta^2}\right)\sum_{i=1}^{m}(2 + r_i)E\left(\frac{z_i^{\alpha}}{(1+z_i^{\alpha})^2}\right)$$

$$E\left(-\frac{\partial^2 l}{\partial\mu\partial\beta}\right) = \left(\frac{\alpha^2}{\beta^2}\right)\sum_{i=1}^{m}(2 + r_i)E\left(\frac{z_i^{\alpha-1}}{(1+z_i^{\alpha})^2}\right)$$

One has to study the estimates and their asymptotic variance-covariance matrix through Monte Calro simulation techniques. Therefore, the observed Fisher information matrix which is given by

$$I_0(\theta) = \begin{bmatrix} -\frac{\partial^2 l}{\partial\mu^2} & -\frac{\partial^2 l}{\partial\mu\partial\beta} \\ -\frac{\partial^2 l}{\partial\beta\partial\mu} & -\frac{\partial^2 l}{\partial\beta^2} \end{bmatrix}_{\mu=\hat{\mu},\beta=\hat{\beta}} \qquad (5.7)$$

Hence, the variance-covariance matrix becomes $I_0^{-1}(\theta)$. Alternatively one can use the equations (5.4-5.6) to obtain observed Fisher information matrix because asymptotically MML estimators and ML estimators are giving same results. The MLEs of reliability $R(t)$ and hazard rate $h(t)$ can be evaluated using invariance properties of MLEs as

$$R(x) = \frac{1}{1+\left(\frac{x-\hat{\mu}}{\hat{\beta}}\right)^{\alpha}};\ x > \hat{\mu}, \alpha > 0, \hat{\beta} > 0 \qquad (5.8)$$

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{\alpha}{\hat{\beta}}\left(\frac{x-\hat{\mu}}{\hat{\beta}}\right)^{\alpha-1}}{1+\left(\frac{x-\hat{\mu}}{\hat{\beta}}\right)^{\alpha}};\ x > \hat{\mu}, \alpha > 0, \hat{\beta} > 0 \qquad (5.9)$$

A two-sided normal approximate confidence interval for the parameters μ and β are

$\hat{\mu} \pm z_{\zeta/2}\sqrt{var_0(\hat{\mu})}$ and $\hat{\beta} \pm z_{\zeta/2}\sqrt{var_0(\hat{\beta})}$,

respectively. Also, the Monte Carlo simulation can be used to find the Coverage Probabilities (CP)

$$CP_{\mu} = P\left[\left|\frac{\hat{\mu}-\mu}{\sqrt{var_0(\hat{\mu})}}\right| \le z_{\zeta/2}\right] \quad \text{and}$$

$$CP_{\beta} = P\left[\left|\frac{\hat{\beta}-\beta}{\sqrt{var_0(\hat{\beta})}}\right| \le z_{\zeta/2}\right] \qquad (5.10)$$

where $z_{\zeta/2}$ is the upper $\zeta/2$ percentile of the standard normal distribution.

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